Infinite series

Series is one of those topics that many students don't find all that useful. To be honest, many students will never see series outside of their calculus class. However, series do play an important role in the field of ordinary differential equations and without series large portions of the field of partial differential equations would not be possible.

In other words, series is an important topic even if you won't ever see any of the applications. Most of the applications are beyond the scope of most Calculus courses and tend to occur in classes that many students don't take. So, as you go through this material keep in mind that these do have applications even if we won't really be covering many of them.

We now consider what happens when we add an infinite number of terms together. Surely if we sum infinitely many numbers, no matter how small they are, the answer goes to infinity? In some cases the answer does indeed go to infinity (like when we sum all the positive integers), but surprisingly there are some cases where the answer is a finite real number.

Sum of an infinite series

1. Cut a piece of string 1 m in length.

2. Now cut the piece of string in half and place one half on the desk.

3. Cut the other half in half again and put one of the pieces on the desk.

4. Repeat this process until the piece of string is too short to cut easily.

5. Draw a diagram to illustrate the sequence of lengths of the pieces of string.

6. Can this sequence be expressed mathematically? Hint: express the shorter lengths of string as a fraction of the original length of string.

7. What is the sum of the lengths of all the pieces of string?

8. Predict what would happen if these steps could be repeated infinitely many times.

9. Will the sum of the lengths of string ever be greater than 1 ?

10. What can you conclude?

1 Convergence and divergence

An infinite series or simply a series is an infinite sum, represented by an infinite expression of the form $a_0 + a_1 + a_2 + \cdots$, where $\{a_n\}$ is any ordered sequence of terms, such as numbers, functions, or anything else that can be added. This is an expression that is obtained from the list of terms $a_0, a_1, \ldots$ by laying them side by side, and conjoining them with the symbol "+". A series may also be represented by using summation notation, such as $\sum_{n=0}^{\infty} a_n$.

If the sum of a series gets closer and closer to a certain value as we increase the number of terms in the sum, we say that the series converges. In other words, there is a limit to the sum of a converging series. If a series does not converge, we say that it diverges. The sum of an infinite series usually tends to infinity, but there are some special cases where it does not.

Given a series $s = \sum_{n=0}^{\infty} a_n$, its kth partial sum is

$s_k = \sum_{n=0}^{k} a_n = a_0 + a_1 + \cdots + a_k.$

By definition, the series $\sum_{n=0}^{\infty} a_n$ converges to the limit L (or simply sums to L), if the sequence of its partial sums has a limit L. In this case, one usually writes $L = \sum_{n=0}^{\infty} a_n$.

A series is said to be convergent if it converges to some limit or divergent when it does not. The value of this limit, if it exists, is then the value of the series.

Note the following:

An arithmetic series never converges: as n tends to infinity, the series will always tend to positive or negative infinity.

Some geometric series converge (have a limit) and some diverge (as n tends to infinity, the series does not tend to any limit or it tends to infinity).

What value does $\left(\frac{2}{5}\right)^n$ approach as n tends towards ∞ ?

$$\begin{aligned}
S_n &= \frac{2}{5} + \frac{4}{25} + \frac{8}{125} + \dots \\
\therefore \text{ first term, } a &= \frac{2}{5} \\
\text{and } \text{ common ratio, } r &= \frac{\frac{4}{25}}{\frac{2}{5}} \\
&= \frac{2}{5} \qquad (-1 < r < 1) \\
\text{So then } S_\infty &= \frac{a}{1-r} \\
&= \frac{\frac{2}{5}}{1-\frac{2}{5}} \\
&= \frac{\frac{2}{5}}{\frac{3}{5}} \\
&= \frac{2}{3}
\end{aligned}$$

2 Comparison Test/Limit Comparison Test

Suppose that we have two series $\sum a_n$ and $\sum b_n$ with $a_n, b_n \geq 0$ for all n and $a_n \leq b_n$ for all n. Then

1. If $\sum b_n$ is convergent then so is $\sum a_n$.
2. If $\sum a_n$ is divergent then so is $\sum b_n$.

Let consider the series $\sum\limits_{n=0}^{\infty} \frac{1}{3^n+n}$.

First, let's note that the series terms are positive.

$\frac{1}{3^n+n} < \frac{1}{3^n}$.

Now, $\sum\limits_{n=0}^{\infty} \frac{1}{3^n}$ is a geometric series and we know that since $|r| = \left|\frac{1}{3}\right| < 1$ the series will converge and its value will be, $\sum\limits_{n=0}^{\infty} \frac{1}{3^n} = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$.

Now, if we go back to our original series and write down the partial sums we get, $s_n = \sum\limits_{i=0}^{n} \frac{1}{3^i+i}$.

Then since $\frac{1}{3^n+n} < \frac{1}{3^n}$, by Comparison test, we can say that $\sum\limits_{n=0}^{\infty} \frac{1}{3^n+n}$ is also convergent.

2.1 Limit Comparison Test

Suppose that we have two series $\sum a_n$ and $\sum b_n$ with $a_n, b_n \geq 0$ for all n. Suppose $c = \lim\limits_{n\to\infty} \frac{a_n}{b_n}$. If c is positive and finite then either both series converge or both series diverge.

3 Ratio Test

Suppose we have the series $\sum a_n$ with $a_n \geq 0$ for all n.
Define, $L = \lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.
Then,
if $L < 1$, the series is convergent.

if $L > 1$, the series is divergent.

if $L = 1$, the ratio test fails.

Let us determine if the following series is convergent or divergent.
$\sum\limits_{n=2}^{\infty} \frac{n^2}{(2n-1)!}$.

$$
\begin{aligned}
L &= \lim_{n\to\infty}\left|\frac{(n+1)^2}{(2(n+1)-1)!}\,\frac{(2n-1)!}{n^2}\right| \\
&= \lim_{n\to\infty}\left|\frac{(n+1)^2}{(2n+1)!}\,\frac{(2n-1)!}{n^2}\right| \\
&= \lim_{n\to\infty}\frac{(n+1)^2}{(2n+1)(2n)(2n-1)!}\,\frac{(2n-1)!}{n^2} \\
&= \lim_{n\to\infty}\frac{(n+1)^2}{(2n+1)(2n)(n^2)} \\
&= 0 < 1
\end{aligned}
$$

So, by the Ratio Test this series converges.

4 Cauchy's nth root test

Suppose that we have the series of positive terms $\sum a_n$.

Define, $L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$.

Then,

if $L < 1$, the series is convergent.

if $L > 1$, the series is divergent.

if $L = 1$, the ratio test fails.

Let us determine whether the series $\sum\limits_{n=1}^{\infty}\left(\frac{1}{3}+\frac{1}{n}\right)^n$ converges or diverges.

Using the root test, we calculate the following limit:
$\lim_{n\to\infty} \sqrt[n]{\left(\frac{1}{3}+\frac{1}{n}\right)^n} = \lim_{n\to\infty} \left(\frac{1}{3}+\frac{1}{n}\right) = \frac{1}{3}+0 = \frac{1}{3} < 1.$
Hence, the given series converges.

5 Integral Test

Suppose that $f(x) > 0$ and is decreasing on the infinite interval $[k, \infty)$ (for some $k \geq 1$) and that $a_n = f(n)$. Then the series $\sum a_n$ converges if and only if the improper integral $\int_1^\infty f(x)\,dx$ converges.

Let us determine if the series $\sum_{n=0}^{\infty} n\mathbf{e}^{-n^2}$ is convergent or divergent.
The function that well use in this example is, $f(x) = x\mathbf{e}^{-x^2}$.
This function is always positive on the interval that we're looking at. Now we need to check that the function is decreasing. It is not clear that this function will always be decreasing on the interval given. The derivative of this function is, $f'(x) = \mathbf{e}^{-x^2}(1-2x^2)$.
This function has two critical points (which will tell us where the derivative changes sign) at $x = \pm 1/\sqrt{2}$. Since we are starting at $n = 0$ we can ignore the negative critical point. Picking a couple of test points we can see that the function is increasing on the interval $[0, 1/\sqrt{2}]$ and it is decreasing on $[1/\sqrt{2}, \infty)$. Therefore, eventually the function will be decreasing and that's all that's required for us to use the Integral Test.

$$\begin{aligned}\int_0^\infty x\mathbf{e}^{-x^2}\,dx &= \lim_{t\to\infty}\int_0^t x\mathbf{e}^{-x^2}\,dx \qquad\qquad u=-x^2\\ &= \lim_{t\to\infty}\left(-\frac{1}{2}\mathbf{e}^{-x^2}\right)\Big|_0^t\\ &= \lim_{t\to\infty}\left(\frac{1}{2}-\frac{1}{2}\mathbf{e}^{-t^2}\right)=\frac{1}{2}\end{aligned}$$

The integral is convergent and so the series must also be convergent by the Integral Test.

6 Alternating series, Leibniz test

An alternating series is any series, $\sum a_n$, for which the series terms can be written in one of the following two forms.

$$a_n = (-1)^n b_n \qquad\qquad b_n \geq 0$$

$$a_n = (-1)^{n+1} b_n \qquad\qquad b_n \geq 0$$

Given a monotonically decreasing sequence a_n which converges to 0, then the alternating series $\sum_{n=1}^{\infty}(-1)^n a_n$ converges. This is the statement of the Leibniz's test.

Let us look at the alternating harmonic series $\sum_{n=1}^{\infty}(-1)^{n-1}1/n$.

In this series, $a_n = 1/n$. Let us check the two conditions.

1. $1/n \geq 1/n + 1$ for all $n \geq 1$.
2. $\lim_{n \to \infty} 1/n = 0$.

Hence, by Leibnitz's test, we conclude that the alternating harmonic series converges.

7 Absolute and Conditional convergence

Sometimes we want to decide whether a series is convergent or divergent, but the sequence isn't necessarily positive. We will learn a technique to evaluate series of this nature but we must first look at a very important definition regarding convergence first.

Let $\sum_{n=1}^{\infty} a_n$ is a convergent series. We say that this series is Absolutely Convergent if $\sum_{n=1}^{\infty} \mid a_n \mid$ is also convergent. We say the original series is Conditionally Convergent if $\sum_{n=1}^{\infty} \mid a_n \mid$ is not convergent.

Theorem 7.1. *If* $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent then it is also convergent.*

Proof. First notice that $|a_n|$ is either a_n or it is $-a_n$ depending on its sign. This means that we can then say,

$0 \leq a_n + |a_n| \leq 2\,|a_n|$.

Now, since we are assuming that $\sum |a_n|$ is convergent then $\sum |2a_n|$ is also convergent since we can just factor the 2 out of the series and 2 times a finite value will still be finite. This however allows us to use the Comparison Test to say that $\sum(a_n + |a_n|)$ is also a convergent series.

Finally, we can write, $\sum a_n = \sum(a_n + |a_n|) - \sum |a_n|$ and so $\sum a_n$ is the difference of two convergent series and so is also convergent. $\square$

From the above theorem, we see that if a series is absolutely convergent then it is convergent, which gives us a test for convergence that does NOT require the series to be positive. Of course, if a series is already positive, then testing for absolute convergence is the same thing as testing for regular convergence. This fact is one of the ways in which absolute convergence is a stronger type of convergence. Series that are absolutely convergent are guaranteed to be convergent. However, series that are convergent may or may not be absolutely convergent.

Let us consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.
This is the alternating harmonic series and we saw earlier that it is a convergent series so we don't need to check that here. So, let's see if it is an absolutely convergent series. To do this we'll need to check the convergence of $\sum\limits_{n=1}^{\infty} \left|\frac{(-1)^n}{n}\right| = \sum\limits_{n=1}^{\infty} \frac{1}{n}$.
This is the harmonic series and we know from the integral test that it is divergent.

Therefore, this series is not absolutely convergent. It is however conditionally convergent since the series itself does converge.

Let us consider another series $\sum_{n=1}^{\infty} \frac{(-1)^{n+2}}{n^2}$.
In this case let's just check absolute convergence first since if it's absolutely convergent we won't need to bother checking convergence as we will get that for free.

$\sum\limits_{n=1}^{\infty} \left| \frac{(-1)^{n+2}}{n^2} \right| = \sum\limits_{n=1}^{\infty} \frac{1}{n^2}$

This series is convergent by the p-series test and so the series is absolute convergent. Note that this does say as well that it's a convergent series.

8 Courtesy

1. https://www.math24.net/ratio-root-tests-page-2/#example4
2. http://tutorial.math.lamar.edu/Classes/CalcII/SeriesIntro.aspx
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4. http://tutorial.math.lamar.edu/Classes/CalcII/AlternatingSeries.aspx
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6. http://mathonline.wikidot.com/absolute-and-conditional-convergence
7. http://tutorial.math.lamar.edu/Classes/CalcII/AbsoluteConvergence.aspx