Power Series Solutions of ODEs

1 Power Series Review

Recall that a power series, about the point $x = c$, is defined as an infinite sum

$$\sum_{k=0}^{\infty} a_k (x-c)^k$$

A power series may only converge for values of x that are near $x = c$. The usual test for convergence is the **ratio test**. The power series converges if

$$|x-c| \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \left| \frac{a_{k+1}(x-c)^{k+1}}{a_k (x-c)^k} \right| < 1$$

If

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = L < \infty$$

then the series converges for $|x-c| < R = 1/L$. Note: the behavior for values of x such that $|x-c| = R$ must be investigated separately.

If the power series converges for $|x-c| < R$ then we can differentiate and integrate term by term without affecting the convergence.

2 Series Solutions of Linear ODEs

If a power series solution of a linear ordinary differential equation exists, and is of the form

$$\sum_{k=0}^{\infty} a_k (x-x_0)^k$$

(where x_0 is the point where initial conditions are specified) then we can substitute the series into the differential equation and try to solve for the coefficients a_k.

As an example, suppose we look for a power series solution of **Airy's Equation**

$$y'' = xy$$

with initial conditions $y(0) = 0, y'(0) = 1$.

If

$$y = \sum_{k=0}^{\infty} a_k x^k$$

then

$$y'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$$

and

$$y''(x) = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}$$

Substituting these into Airy's equation gives

$$\sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} = x \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_k x^{k+1}$$

We cannot compare these series directly, so we make a change of index $m = k - 2$ for the left sum, and $m = k + 1$ for the right sum

$$\sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m = \sum_{m=1}^{\infty} a_{m-1} x^m$$

These series begin and different indices, so we rewrite the left sum as

$$(2)(1)a_2 + \sum_{m=1}^{\infty} (m+2)(m+1) a_{m+2} x^m = \sum_{m=1}^{\infty} a_{m-1} x^m$$

Since the coefficients of two equivalent power series must be identical, we have the conditions

$$a_2 = 0$$

and

$$(m+2)(m+1) a_{m+2} = a_{m-1}$$

The latter can be written as

$$a_{m+2} = \frac{a_{m-1}}{(m+2)(m+1)}$$

which is called the **recurrence relation**.

The recurrence relation determines the values of all the coefficients in terms of the first few coefficients. Specifically,

For $m = 1, 4, 7, \ldots$: $a_3 = \frac{a_0}{3*2}, a_6 = \frac{a_3}{6*5} = \frac{1}{6*5}\frac{a_0}{3*2}, a_9 = \ldots.$

For $m = 2, 5, 8, \ldots$: $a_4 = \frac{a_1}{4*3}, a_7 = \frac{a_4}{7*6} = \frac{1}{7*6}\frac{a_1}{4*3}, a_{10} = \ldots.$

For $m = 3, 6, 9, \ldots$: $a_5 = \frac{a_2}{5*4} = 0, a_8 = \frac{a_5}{8*7} = 0, a_{11} = 0\ldots.$

If we subsitute these coefficients back into the power series, we get

$$y(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

$$a_0 + a_1 x + \frac{a_0}{3*2}x^3 + \frac{a_1}{4*3}x^4 + \frac{a_0}{6*5*3*2}x^6 + \frac{a_1}{7*6*4*3}x^7 + ...$$
$$= a_0 \left[1 + \frac{1}{3*2}x^3 + \frac{1}{6*5*3*2}x^6 + ...\right] + a_1 \left[x + \frac{1}{4*3}x^4 + \frac{1}{7*6*4*3}x^7 + ...\right]$$
$$= a_0 y_1(x) + a_2 y_2(x)$$

The functions $y_1(x)$ and $y_2(x)$ are the Airy functions of the first and second kind respectively. Note, if $a_2 \neq 0$ there would be a **third** linearly independent solution of Airy's equation, which cannot happen. The restriction on two solutions forces one third of the coefficients to vanish!

3 Using the Differential Equation to Construct the Series Term by Term

The differential equation

$$y'' = xy$$

actually provides a recipe for reconstructing the solutions of Airy's equation **term by term**.

Begin with the initial conditions $y(0) = 1$, $y'(0) = 0$. The differential equation gives us $y''(0) = 0y(0) = 0$, Differentiating the ODE gives $y'''(x) = y(x) + xy'(x)$ which implies that $y'''(0) = y(0) = 1$. Assuming the solution has a convergent Taylor series expansion we have

$$y(x) = y(0) + y'(0)x + y''(0)\frac{x^2}{2!} + ...$$
$$= 1 + 0x + 0x^2 + 1\frac{x^3}{3!} + ...$$
$$= y_1(x)$$

Unlike the recurrence relation, we don't have a formula for the coefficients in terms of the initial terms, we must calculate derivatives successively.

The big advantage of the term-by-term method is that it can be used for nonlinear ODEs!

For example, consider the initial value problem

$$y'(x) = y^2(x), y(0) = 1$$

Using the initial conditions, and successive differentiation of the nonlnear ODE, we get $y(0) = 1$, $y'(0) = y(0)^2 = 1$, $y''(x) = 2y(x)y'(x) \Rightarrow y''(0) = 2y(0)y'(0) = 2$, $y'''(x) = 2y'(x)^2 + 2y(x)y''(x) \Rightarrow y'''(0) = 2y'(0)^2 + 2y(0)y''(0) = 6$, etc.

Constructing the Taylor series, we get

$$y(x) = y(0) + y'(0)x + y''(0)x^2/2! + ...$$
$$= 1 + x + x^2 + x^3 + ...$$

In fact the coefficient of all the terms is 1, which leads to

$$y(x) = 1 + x + x^2 + x^3 + ... + x^n + ... = \frac{1}{1-x}$$

which is in fact the **solution** of $y' = y^2, y(0) = 1$!

If you try to get recurrence relation from this nonline ODE, you must find a way to **square** the power series, that is, you need to compute

$$(\sum_{n=0}^{\infty} a_n (x-c)^n)^2$$

which is possible but leads to very messy (and nonlinear) quantities involving the coefficients!

3 Vectors: Triple Products

3.1 The Scalar Triple Product

The scalar triple product, as its name may suggest, results in a scalar as its result. It is a means of combining three vectors via cross product and a dot product. Given the vectors

$$\begin{aligned}\mathbf{A} &= A_1\,\mathbf{i} + A_2\,\mathbf{j} + A_3\,\mathbf{k}\\ \mathbf{B} &= B_1\,\mathbf{i} + B_2\,\mathbf{j} + B_3\,\mathbf{k}\\ \mathbf{C} &= C_1\,\mathbf{i} + C_2\,\mathbf{j} + C_3\,\mathbf{k}\end{aligned}$$

a scalar triple product will involve a dot product and a cross product

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$$

It is necessary to perform the cross product before the dot product when computing a scalar triple product,

$$\mathbf{B}\times\mathbf{C} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ B_1 & B_2 & B_3\\ C_1 & C_2 & C_3\end{vmatrix} = \mathbf{i}\begin{vmatrix}B_2 & B_3\\ C_2 & C_3\end{vmatrix} - \mathbf{j}\begin{vmatrix}B_1 & B_3\\ C_1 & C_3\end{vmatrix} + \mathbf{k}\begin{vmatrix}B_1 & B_2\\ C_1 & C_2\end{vmatrix}$$

since $\mathbf{A} = A_1\,\mathbf{i} + A_2\,\mathbf{j} + A_3\,\mathbf{k}$ one can take the dot product to find that

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = (A_1)\begin{vmatrix}B_2 & B_3\\ C_2 & C_3\end{vmatrix} - (A_2)\begin{vmatrix}B_1 & B_3\\ C_1 & C_3\end{vmatrix} + (A_3)\begin{vmatrix}B_1 & B_2\\ C_1 & C_2\end{vmatrix}$$

which is simply

Important Formula 3.1.

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix}A_1 & A_2 & A_3\\ B_1 & B_2 & B_3\\ C_1 & C_2 & C_3\end{vmatrix}$$

The usefulness of being able to write the scalar triple product as a determinant is not only due to convenience in calculation but also due to the following property of determinants

Note 3.1. Exchanging any two adjacent rows in a determinant changes the sign of the original determinant.

Thus,

$$\mathbf{B}\cdot(\mathbf{A}\times\mathbf{C}) = \begin{vmatrix} B_1 & B_2 & B_3 \\ A_1 & A_2 & A_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = -\begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = -\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}).$$

Formula 3.1.

$$\mathbf{B}\cdot(\mathbf{A}\times\mathbf{C}) = -\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}).$$

3.1.1 Worked examples.

Example 3.1.1. Given,

$$\begin{aligned} \mathbf{A} &= 2\,\mathbf{i} + 3\,\mathbf{j} - 1\,\mathbf{k} \\ \mathbf{B} &= -\,\mathbf{i} + \mathbf{j} \\ \mathbf{C} &= 2\,\mathbf{i} + 2\,\mathbf{j} \end{aligned}$$

Find

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$$

Solution:
Method 1:

Begin by finding

$$\begin{aligned} \mathbf{B}\times\mathbf{C} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 0 \\ 2 & 2 & 0 \end{vmatrix} &= \mathbf{i}\begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix} - \mathbf{j}\begin{vmatrix} -1 & 0 \\ 2 & 0 \end{vmatrix} + \mathbf{k}\begin{vmatrix} -1 & 1 \\ 2 & 2 \end{vmatrix} \\ &= \big((1)(0)-(0)(2)\big)\mathbf{i} - \big((-1)(0)-(0)(2)\big)\mathbf{j} + \big((-1)(2)-(1)(2)\big)\mathbf{k} \\ &= 0\,\mathbf{i} + 0\,\mathbf{j} - 4\,\mathbf{k}. \end{aligned}$$

... example continued

Take the dot product with **A** to find

$$\begin{aligned}\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) &= (2)(0)+(3)(0)+(-1)(-4)\\ &= 4\end{aligned}$$

Method 2:
Evaluate the determinant

$$\begin{aligned}\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) &= \begin{vmatrix} 2 & 3 & -1\\ -1 & 1 & 0\\ 2 & 2 & 0\end{vmatrix} = (2)\begin{vmatrix}1 & 0\\ 2 & 0\end{vmatrix} - (3)\begin{vmatrix}-1 & 0\\ 2 & 0\end{vmatrix} + (-1)\begin{vmatrix}-1 & 1\\ 2 & 2\end{vmatrix}\\ &= (2)\big((1)(0)-(0)(0)\big) - (3)\big((-1)(0)-(0)(2)\big) + (-1)\big((-1)(2)-(1)(2)\big)\\ &= 4\end{aligned}$$

Example 3.1.2. Prove that

Important Formula 3.2.

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \mathbf{A}\times\mathbf{B}\cdot\mathbf{C}$$

Solution:
Notice that there are no brackets given here as the only way to evaluate the scalar triple products is to perform the cross products before performing the dot products[a]. Let

$$\begin{aligned}\mathbf{A} &= A_1\,\mathbf{i} + A_2\,\mathbf{j} + A_3\,\mathbf{k}\\ \mathbf{B} &= B_1\,\mathbf{i} + B_2\,\mathbf{j} + B_3\,\mathbf{k}\\ \mathbf{C} &= C_1\,\mathbf{i} + C_2\,\mathbf{j} + C_3\,\mathbf{k}\end{aligned}$$

now,

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \begin{vmatrix}A_1 & A_2 & A_3\\ B_1 & B_2 & B_3\\ C_1 & C_2 & C_3\end{vmatrix} = -\begin{vmatrix}C_1 & C_2 & C_3\\ B_1 & B_2 & B_3\\ A_1 & A_2 & A_3\end{vmatrix} = \begin{vmatrix}C_1 & C_2 & C_3\\ A_1 & A_2 & A_3\\ B_1 & B_2 & B_3\end{vmatrix} = \mathbf{C}\cdot\mathbf{A}\times\mathbf{B} = \mathbf{A}\times\mathbf{B}\cdot\mathbf{C}$$

[a] This is due to the fact that if the dot product is evaluate first one would be left with a cross product between a scalar and a vector which is not defined.

3.2 The Vector Triple Product

The vector triple product, as its name suggests, produces a vector. It is the result of taking the cross product of one vector with the cross product of two other vectors.

Important Formula 3.3 (Vector Triple Product)**.**

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$$

Proving the vector triple product formula can be done in a number of ways. The straightforward method is to assign

$$\begin{aligned} \mathbf{A} &= A_1\,\mathbf{i} + A_2\,\mathbf{j} + A_3\,\mathbf{k} \\ \mathbf{B} &= B_1\,\mathbf{i} + B_2\,\mathbf{j} + B_3\,\mathbf{k} \\ \mathbf{C} &= C_1\,\mathbf{i} + C_2\,\mathbf{j} + C_3\,\mathbf{k} \end{aligned}$$

and work out the various dot and cross products to show that the result is the same. Here we shall however go through a slightly more subtle but less calculation heavy proof.

Note 3.2. The vector $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ must be in the same plane as $\mathbf{B}$ and $\mathbf{C}$. This is due to fact that the vector that results from the cross product is perpendicular to both the vectors whose product has just been taken. Since one can say that $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ is on the same plane as $\mathbf{B}$ and $\mathbf{C}$ it follows that

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \alpha\mathbf{B} + \beta\mathbf{C}$$

where α and β are scalars.

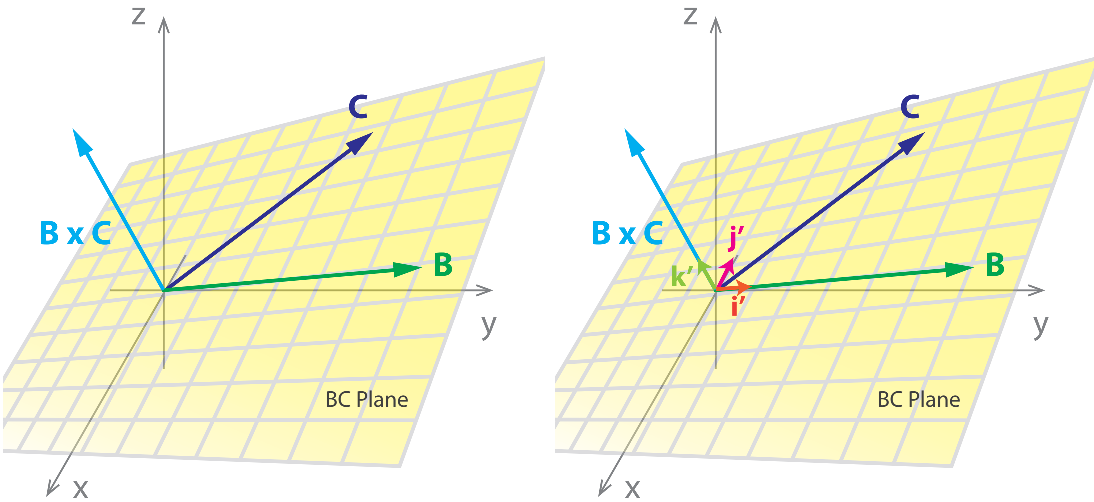
We introduce a new coordinate system with the unit vector $\mathbf{i}'$ along the vector $\mathbf{B}$, $\mathbf{j}'$ a unit vector (orthogonal to $\mathbf{i}'$), which is on the same plane as the both the vectors $\mathbf{B}$ and $\mathbf{C}$, and $\mathbf{k}'$ a unit vector orthogonal to both $\mathbf{i}'$ and $\mathbf{j}'$[1]. Using this *basis* allows one to write the vectors $\mathbf{B}$ and $\mathbf{C}$ as

$$\begin{aligned} \mathbf{B} &= B_1\mathbf{i}' + 0\mathbf{j}' + 0\mathbf{k}' \\ \mathbf{C} &= C_1\mathbf{i}' + C_2\mathbf{j}' + 0\mathbf{k}' \end{aligned}$$

however there is no special reduction to the representation of the vector $\mathbf{A}$ in terms of this new basis thus,

$$\mathbf{A} = A_1\mathbf{i}' + A_2\mathbf{j}' + A_3\mathbf{k}'.$$

[1] the unit vector $\mathbf{k}'$ will thus point in the same direction as the vector $\mathbf{B} \times \mathbf{C}$.



(a) The vectors **B** and **C** define the BC-plane.

(b) The unit vectors $\mathbf{i}'$ and $\mathbf{j}'$ are on the BC-plane while $\mathbf{k}'$ points in the same direction as $(\mathbf{B}\times\mathbf{C})$

Figure 1: Figures representing the change in basis.

We know that the vector $\mathbf{B}\times\mathbf{C}$ must be of the form of $0\mathbf{i}'+0\mathbf{j}'+\gamma\mathbf{k}'$ for some scalar γ. We find the value of γ by taking the cross product

$$\mathbf{B}\times\mathbf{C} = \begin{vmatrix} \mathbf{i}' & \mathbf{j}' & \mathbf{k}' \\ B_1 & 0 & 0 \\ C_1 & C_2 & 0 \end{vmatrix} = \mathbf{i}'\begin{vmatrix} 0 & 0 \\ C_2 & 0 \end{vmatrix} - \mathbf{j}'\begin{vmatrix} B_1 & 0 \\ C_1 & 0 \end{vmatrix} + \mathbf{k}'\begin{vmatrix} B_1 & 0 \\ C_1 & C_2 \end{vmatrix} = 0\mathbf{i}' + 0\mathbf{j}' + B_1C_2\mathbf{k}'$$

We have now found that

$$\mathbf{B}\times\mathbf{C} = B_1C_2\mathbf{k}'$$

.

Now examining the final cross product

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} \mathbf{i}' & \mathbf{j}' & \mathbf{k}' \\ A_1 & A_2 & A_3 \\ 0 & 0 & B_1C_2 \end{vmatrix} = A_2B_1C_2\mathbf{i}' - A_1B_1C_2\mathbf{j}' + 0\mathbf{k}'.$$

thus,

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = A_2B_1C_2\mathbf{i}' - A_1B_1C_2\mathbf{j}'$$

.

Here a clever addition of zero is useful

$$\begin{aligned}\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) &= A_2B_1C_2\mathbf{i}' - A_1B_1C_2\mathbf{j}' \underbrace{+A_1B_1C_1\mathbf{i}' - A_1B_1C_1\mathbf{i}'}_{0} \\ &= (A_2C_2 + A_1C_1)B_1\mathbf{i}' - A_1B_1(C_1\mathbf{i}' + C_2\mathbf{j}').\end{aligned}$$

This is the desired result as returning to our definitions of $\mathbf{A}$, $\mathbf{B}$and $\mathbf{C}$ in this basis,

$$\begin{aligned}
\mathbf{A} &= A_1\mathbf{i}' + A_2\mathbf{j}' + A_3\mathbf{k}' \\
\mathbf{B} &= B_1\mathbf{i}' \\
\mathbf{C} &= C_1\mathbf{i}' + C_2\mathbf{j}'
\end{aligned}$$

one finds that,

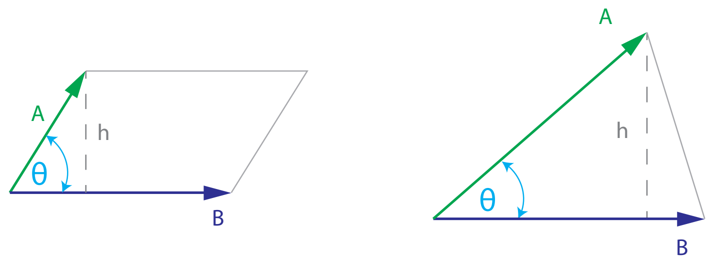
$$\begin{aligned}
\mathbf{A} \cdot \mathbf{B} &= A_1 B_1 \\
\mathbf{A} \cdot \mathbf{C} &= A_1 C_1 + A_2 C_2
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= (A_2 C_2 + A_1 C_1) B_1 \mathbf{i}' - A_1 B_1 (C_1 \mathbf{i}' + C_2 \mathbf{j}') \\
&= (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}.
\end{aligned}$$

3.3 Area and Volume Using a Cross Product

3.3.1 The area of a parallelogram.



(a) The parallelogram (b) The triangle

Figure 2: Areas related to the cross product.

The are of a parallelogram is simply given by the product of the base and the height of the parallelogram. Here this is given by

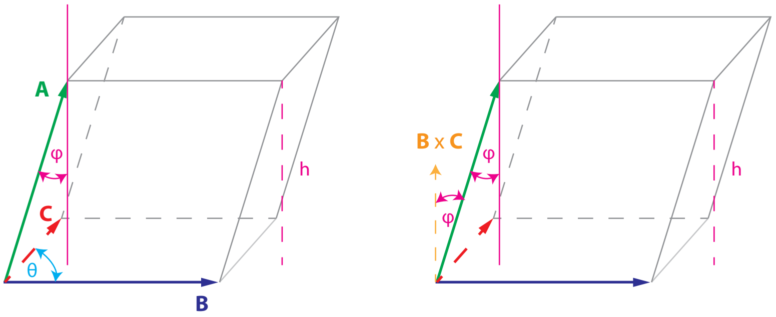
$$\begin{aligned}
\text{Area of parallelogram} &= h\,|\,\mathbf{B}\,| \\
&= (|\,\mathbf{A}\,|\sin(\theta))\,|\,\mathbf{B}\,| \\
&= |\,\mathbf{A}\,|\,|\,\mathbf{B}\,|\sin(\theta) \\
&= |\mathbf{A}\times\mathbf{B}|
\end{aligned}$$

3.3.2 The area of a triangle.

The area of a triangle is half the base times the height. From the figure we have

$$\begin{aligned}
\text{Area of triangle} &= \frac{1}{2}\,|\,\mathbf{B}\,|\,h = \frac{1}{2}\,|\,\mathbf{B}\,|\,(|\,\mathbf{A}\,|\sin(\theta)) \\
&= \frac{1}{2}\,|\mathbf{A}\times\mathbf{B}|
\end{aligned}$$

3.3.3 The volume of a parallelepiped.



(a) The angle between **B** and **C** is θ.

(b) The angle between h and **A** is ϕ.

Figure 3: A parallelepiped.

ADVANCED ASIDE 3.1. A parallelepiped is a three dimensional object whose six sides are parallelograms. The volume of a parallelepiped is given by

$$V = (\text{Base Area})(\text{Height})$$

The area of the base is the area of a parallelogram as such one has

$$\text{Area of the Base} = |\mathbf{B} \times \mathbf{C}|$$

The height h requires a little geometry but is simply

$$h = |\,\mathbf{A}\,| \cos(\phi)$$

notice that the vector $\mathbf{B} \times \mathbf{A}$ is parallel to the line h. Thus the vector **A** makes an angle[a] with the vector $\mathbf{B} \times \mathbf{C}$ of $\gamma = \phi$. Finally we have the volume of the parallelepiped given by

$$\begin{aligned}
\text{Volume of parallelepiped} &= (\text{Base})(\text{height}) \\
&= (|\mathbf{B} \times \mathbf{C}|)\,(|\,\mathbf{A}\,|\,|\cos(\gamma)|) \\
&= |\,\mathbf{A}\,|\,|\mathbf{B} \times \mathbf{C}|\,|\cos(\gamma)| \\
&= |\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})|
\end{aligned}$$

[a] It is also possible for $\mathbf{B} \times \mathbf{C}$ to make an angle $\gamma = 180° - \phi$ which does not affect the result since $|\cos(180° - \phi)| = |\cos(\phi)|$

3.4 Summary of Vector Rules

Here we list most of the main results concerning vectors,

$$\mathbf{A}\cdot\mathbf{A} = A^2 \equiv |\,\mathbf{A}\,|^2 \qquad (3.4.1)$$

$$\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A} \qquad (3.4.2)$$

$$\mathbf{A}\cdot(\alpha\mathbf{B}) = \alpha(\mathbf{A}\cdot\mathbf{B}) \qquad (3.4.3)$$

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\times\mathbf{B})\cdot\mathbf{C} \qquad (3.4.4)$$

$$\mathbf{A}\times\mathbf{A} = \mathbf{0} \qquad (3.4.5)$$

$$\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A} \qquad (3.4.6)$$

$$\mathbf{A}\times(\alpha\mathbf{B}) = \alpha(\mathbf{A}\times\mathbf{B}) \qquad (3.4.7)$$

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C} \qquad (3.4.8)$$

3.4.1 Worked problem

Example 3.4.1 (Manipulating vectors without evaluation).
Prove that

(i) $(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D}) = \mathbf{C}(\mathbf{A}\cdot\mathbf{B}\times\mathbf{D}) - \mathbf{D}(\mathbf{A}\cdot\mathbf{B}\times\mathbf{C})$

(ii) $(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}\times\mathbf{D}) - \mathbf{A}(\mathbf{B}\cdot\mathbf{C}\times\mathbf{D})$

Solution:

(i) let $\mathbf{U} = \mathbf{A}\times\mathbf{B}$

$$\begin{aligned}
(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D}) &= \mathbf{U}\times(\mathbf{C}\times\mathbf{D}) \\
&= (\mathbf{U}\cdot\mathbf{D})\mathbf{C} - (\mathbf{U}\cdot\mathbf{C})\mathbf{D} && \text{using Eq 3.4.8} \\
&= (\mathbf{A}\times\mathbf{B}\cdot\mathbf{D})\mathbf{C} - (\mathbf{A}\times\mathbf{B}\cdot\mathbf{C})\mathbf{D} \\
&= (\mathbf{A}\cdot\mathbf{B}\times\mathbf{D})\mathbf{C} - (\mathbf{A}\cdot\mathbf{B}\times\mathbf{C})\mathbf{D} && \text{using Eq 3.4.4} \\
&= \mathbf{C}(\mathbf{A}\cdot\mathbf{B}\times\mathbf{D}) - \mathbf{D}(\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}).
\end{aligned}$$

(ii) let $\mathbf{V} = (\mathbf{C}\times\mathbf{D})$

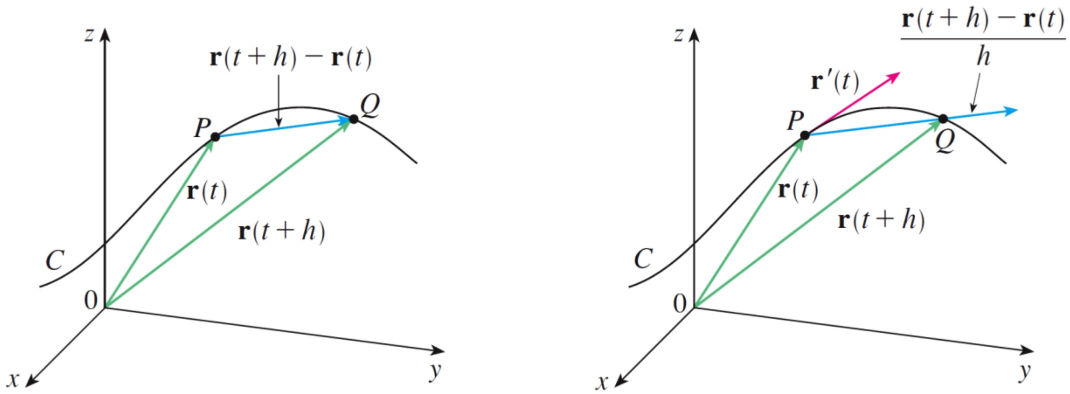
$$\begin{aligned}
(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D}) &= -(\mathbf{C}\times\mathbf{D})\times(\mathbf{A}\times\mathbf{B}) && \text{using Eq 3.4.6} \\
&= -\mathbf{V}\times(\mathbf{A}\times\mathbf{B}) \\
&= -\{(\mathbf{V}\cdot\mathbf{B})\mathbf{A} - (\mathbf{V}\cdot\mathbf{A})\mathbf{B}\} && \text{using Eq 3.4.8} \\
&= (\mathbf{C}\times\mathbf{D}\cdot\mathbf{A})\mathbf{B} - (\mathbf{C}\times\mathbf{D}\cdot\mathbf{B})\mathbf{A} \\
&= (\mathbf{C}\cdot\mathbf{D}\times\mathbf{A})\mathbf{B} - (\mathbf{C}\cdot\mathbf{D}\times\mathbf{B})\mathbf{A} && \text{using Eq 3.4.4} \\
&= \mathbf{B}(\mathbf{A}\cdot\mathbf{C}\times\mathbf{D}) - \mathbf{A}(\mathbf{B}\cdot\mathbf{C}\times\mathbf{D}).
\end{aligned}$$

Derivatives

The derivative $\mathbf{r}'$ of a vector function $\mathbf{r}$ is defined in much the same way as for real-valued functions:

1
$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

if this limit exists. The geometric significance of this definition is shown in Figure 1.



(a) The secant vector $\overrightarrow{PQ}$ (b) The tangent vector $\mathbf{r}'(t)$

Figure 1

Derivatives

If the points P and Q have position vectors $\mathbf{r}(t)$ and $\mathbf{r}(t + h)$, then $\overrightarrow{PQ}$ represents the vector $\mathbf{r}(t + h) - \mathbf{r}(t)$, which can therefore be regarded as a secant vector.

If $h > 0$, the scalar multiple $(1/h)(\mathbf{r}(t + h) - \mathbf{r}(t))$ has the same direction as $\mathbf{r}(t + h) - \mathbf{r}(t)$. As $h \to 0$, it appears that this vector approaches a vector that lies on the tangent line.

For this reason, the vector $\mathbf{r}'(t)$ is called the **tangent vector** to the curve defined by $\mathbf{r}$ at the point P, provided that $\mathbf{r}'(t)$ exists and $\mathbf{r}'(t) \neq \mathbf{0}$.

Derivatives

The **tangent line** to *C* at *P* is defined to be the line through *P* parallel to the tangent vector $\mathbf{r}'(t)$.

We will also have occasion to consider the **unit tangent vector**, which is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

Derivatives

The following theorem gives us a convenient method for computing the derivative of a vector function **r**: just differentiate each component of **r**.

2 Theorem If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$, where f, g, and h are differentiable functions, then

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle = f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j} + h'(t)\,\mathbf{k}$$

Example 1

(a) Find the derivative of $\mathbf{r}(t) = (1 + t^3)\mathbf{i} + te^{-t}\,\mathbf{j} + \sin 2t\,\mathbf{k}$.

(b) Find the unit tangent vector at the point where $t = 0$.

Solution:

(a) According to Theorem 2, we differentiate each component of $\mathbf{r}$:

$$\mathbf{r}'(t) = 3t^2\,\mathbf{i} + (1 - t)e^{-t}\,\mathbf{j} + 2\cos 2t\,\mathbf{k}$$

Example 1 – *Solution*

cont'd

(b) Since **r**(0) = **i** and **r**′(0) = **j** + 2**k**, the unit tangent vector at the point (1, 0, 0) is

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|}$$

$$= \frac{\mathbf{j} + 2\mathbf{k}}{\sqrt{1 + 4}}$$

$$= \frac{1}{\sqrt{5}}\mathbf{j} + \frac{2}{\sqrt{5}}\mathbf{k}$$

Derivatives

Just as for real-valued functions, the **second derivative** of a vector function $\mathbf{r}$ is the derivative of $\mathbf{r}'$, that is, $\mathbf{r}'' = (\mathbf{r}')'$.

For instance, the second derivative of the function, $\mathbf{r}(t) = \langle 2 \cos t, \sin t, t \rangle$, is

$$\mathbf{r}''(t) = \langle -2 \cos t, -\sin t, 0 \rangle$$

Differentiation Rules

The next theorem shows that the differentiation formulas for real-valued functions have their counterparts for vector-valued functions.

3 Theorem Suppose $\mathbf{u}$ and $\mathbf{v}$ are differentiable vector functions, c is a scalar, and f is a real-valued function. Then

1. $\dfrac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$

2. $\dfrac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$

3. $\dfrac{d}{dt}[f(t)\,\mathbf{u}(t)] = f'(t)\,\mathbf{u}(t) + f(t)\,\mathbf{u}'(t)$

4. $\dfrac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$

5. $\dfrac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$

6. $\dfrac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\,\mathbf{u}'(f(t))$ (Chain Rule)

Example 4

Show that if $|\mathbf{r}(t)| = c$ (a constant), then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$ for all t.

Solution:

Since

$$\mathbf{r}(t) \cdot \mathbf{r}(t) = |\mathbf{r}(t)|^2 = c^2$$

and c^2 is a constant, Formula 4 of Theorem 3 gives

$$0 = \frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{r}(t)] = \mathbf{r}'(t) \cdot \mathbf{r}(t) + \mathbf{r}(t) \cdot \mathbf{r}'(t) = 2\mathbf{r}'(t) \cdot \mathbf{r}(t)$$

Thus $\mathbf{r}'(t) \cdot \mathbf{r}(t) = 0$, which says that $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$.

Example 4 – *Solution*

cont'd

Geometrically, this result says that if a curve lies on a sphere with center the origin, then the tangent vector $\mathbf{r}'(t)$ is always perpendicular to the position vector $\mathbf{r}(t)$. (See Figure 4.)

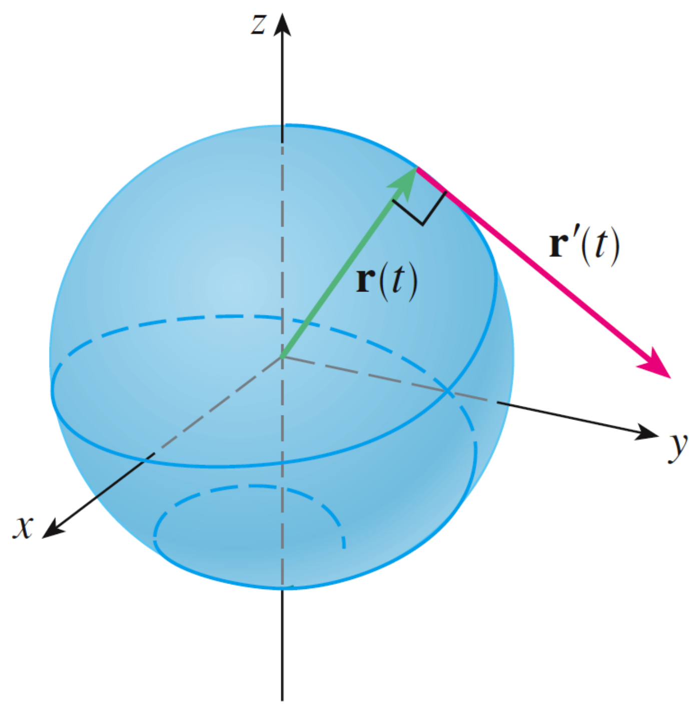


Figure 4

Integrals

The **definite integral** of a continuous vector function **r**(*t*) can be defined in much the same way as for real-valued functions except that the integral is a vector.

But then we can express the integral of **r** in terms of the integrals of its component functions *f*, *g*, and *h* as follows.

$$\int_a^b \mathbf{r}(t)\,dt = \lim_{n\to\infty} \sum_{i=1}^{n} \mathbf{r}(t_i^*)\,\Delta t$$

$$= \lim_{n\to\infty} \left[\left(\sum_{i=1}^{n} f(t_i^*)\,\Delta t \right) \mathbf{i} + \left(\sum_{i=1}^{n} g(t_i^*)\,\Delta t \right) \mathbf{j} + \left(\sum_{i=1}^{n} h(t_i^*)\,\Delta t \right) \mathbf{k} \right]$$

Integrals

So

$$\int_a^b \mathbf{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\mathbf{i} + \left(\int_a^b g(t)\,dt\right)\mathbf{j} + \left(\int_a^b h(t)\,dt\right)\mathbf{k}$$

This means that we can evaluate an integral of a vector function by integrating each component function.

Integrals

We can extend the Fundamental Theorem of Calculus to continuous vector functions as follows:

$$\int_a^b \mathbf{r}(t)\,dt = \mathbf{R}(t)\Big]_a^b = \mathbf{R}(b) - \mathbf{R}(a)$$

where **R** is an antiderivative of **r**, that is, $\mathbf{R}'(t) = \mathbf{r}(t)$.

We use the notation $\int \mathbf{r}(t)\,dt$ for indefinite integrals (antiderivatives).

Example 5

If $\mathbf{r}(t) = 2 \cos t\, \mathbf{i} + \sin t\, \mathbf{j} + 2t\, \mathbf{k}$, then

$$\int \mathbf{r}(t)\, dt = \left(\int 2 \cos t\, dt \right) \mathbf{i} + \left(\int \sin t\, dt \right) \mathbf{j} + \left(\int 2t\, dt \right) \mathbf{k}$$

$$= 2 \sin t\, \mathbf{i} - \cos t\, \mathbf{j} + t^2\, \mathbf{k} + \mathbf{C}$$

where $\mathbf{C}$ is a vector constant of integration, and

$$\int_0^{\pi/2} \mathbf{r}(t)\, dt = \left[2 \sin t\, \mathbf{i} - \cos t\, \mathbf{j} + t^2\, \mathbf{k}\right]_0^{\pi/2}$$

$$= 2\mathbf{i} + \mathbf{j} + \frac{\pi^2}{4}\mathbf{k}$$

Courtesy : ---

1. Course Notes for Math 308H - Spring 2016) by Dr. Michael S. Pilant
   April 27, 2016

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