➤ 1.3 Coulomb's Law

Coulomb's Law gives the force of attraction (unlike charges) or repulsion (like charges) between two point charges. Two point charges q_1 and q_2 separated by distance r exert on each other forces along the line joining them which is proportional to the product of the charges and inversely proportional to the separation squared.

Thus,
$$F = k\frac{q_1 q_2}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}$$

The constant ε_0 is called permittivity of free space.

If instead of free space (air), the charges are situated in some medium, the force between them can be written as

$$F_{med} = \frac{1}{4\pi\varepsilon}\frac{q_1 q_2}{r^2} = \frac{1}{4\pi\varepsilon_0 k}\frac{q_1 q_2}{r^2};$$

where, $\varepsilon = \varepsilon_0 k$ = permittivity of the medium.

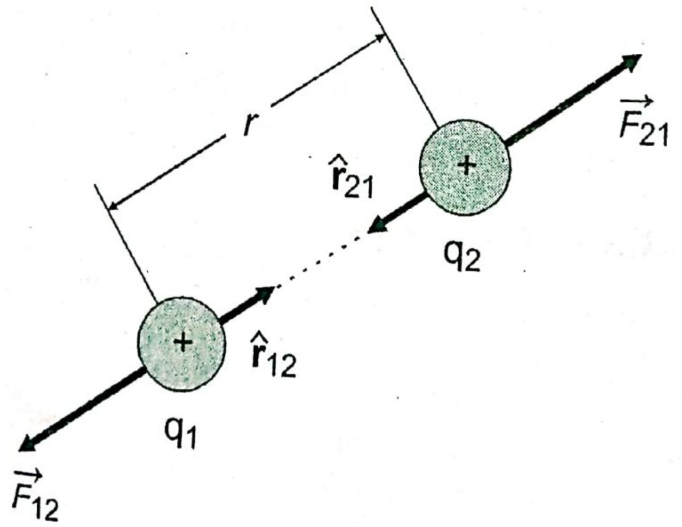
Here, k is the relative permittivity or dielectric constant of the medium.

$$\frac{F_{med}}{F_{air}} = \frac{\frac{1}{4\pi\varepsilon_0 k}\frac{q_1 q_2}{r^2}}{\frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}} = \frac{1}{k} \Rightarrow k = \frac{F_{air}}{F_{med}}$$

Thus, dielectric constant,

$$k = \frac{\text{force between two charges in air}}{\text{force between same charges at same separation in that medium}}$$

We can write the Coulomb's law in vector form as follows :



The force on charge q_1 due to q_2, i.e., $\vec{F}_{12} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{21}^2}\hat{r}_{21}$

$\hat{r}_{21}$ is the unit vector from q_2 to q_1 and $r_{21} = |\vec{r}_2 - \vec{r}_1|$ is the distance from q_2 to q_1.

In a similar way, the force on charge q_2 due to q_1 is given by

$$\vec{F}_{21} = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r_{21}^2} \hat{r}_{12} = -\vec{F}_{12}$$

Thus the force on one charge due to second is equal and opposite to the force on second due to the first.

Important points : (a) The charges in the Coulomb's law are assumed to be static. If the charges are moving, they will give rise to magnetic field in addition to electrostatic field.

(b) The charges must be point charge, otherwise the distance between them cannot be specified uniquely.

➤ 1.4 Principle of Superposition

When several point charges are present, the total force on a particular charge q is the vector sum of the individual forces due to each of the other charges. Thus, electric forces have a superposition property.

According to the **principle of superposition,** the force on any charge due to a number of other charges is equal to the vector sum of all the forces on that charge due to other charges. The force between any two charges is unaffected by the presence of other charges.

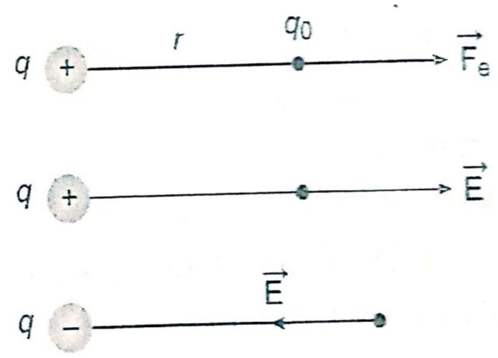
Thus for a system of n charges $q_1, q_2, \ldots, q_n$, the total force $\vec{F}_1$ on charge q_1 due to all other charges is given by

$$\vec{F}_1 = \vec{F}_{12} + \vec{F}_{13} + \ldots + \vec{F}_{1n} = \sum_{i=2}^{n} \vec{F}_{1i} = \frac{1}{4\pi\varepsilon_0} \sum_{i=2}^{n} \frac{q_1 q_i}{(r_1 - r_i)^2} \hat{r}_{1i}$$

In the Summation $i = 1$ is omitted, because a charge can not exert force on itself.

➤ 1.5 Electric Field

A charged particle at rest produces electric field. But if the charged particle is in uniform (unaccelerated) motion, it produces both electric and magnetic fields. An accelerated charged particle not only produces an electric and magnetic fields, but also radiates energy in the form of electromagnetic waves.



The electric field intensity (or electric field strength) $\vec{E}$ is the force per unit charge when placed in the electric field.

Mathematically,
$$\vec{E} = \lim_{q_0 \to 0} \frac{\vec{F}_e}{q_0}$$

Here, limit $q_0 \to 0$ is taken to ensure that the test charge is so small that it does not produce its own electric field and disturb the source electric field in the region. The direction of electric field intensity $\vec{E}$ is obviously in the direction of the force $\vec{F}_e$ and is expressed in newtons/coulomb or volts/meter. If q is positive, $\vec{E}$ is directed away from q. On the other hand if q is negative, then $\vec{E}$ is directed towards q.

The electric field intensity at point $\vec{r}$ due to a point charge Q located at $\vec{r}'$ is given by

$$\vec{E} = \frac{\vec{F}_e}{q_0} = \frac{1}{q_0}\frac{Qq_0}{4\pi\varepsilon_0}\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} = \frac{Q}{4\pi\varepsilon_0}\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

For N point charges $Q_1, Q_2, \ldots, Q_N$ located at $\vec{r}_1, \vec{r}_2 \ldots \vec{r}_N$, the electric field intensity at point $\vec{r}$ is readily obtained as

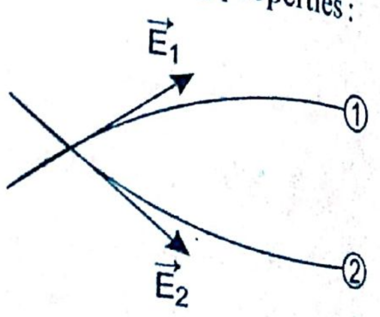
$$\vec{E} = \frac{Q_1}{4\pi\varepsilon_0}\frac{(\vec{r}-\vec{r}_1)}{|\vec{r}-\vec{r}_1|^3} + \frac{Q_2}{4\pi\varepsilon_0}\frac{(\vec{r}-\vec{r}_2)}{|\vec{r}-\vec{r}_2|^3} + \ldots + \frac{Q_N}{4\pi\varepsilon_0}\frac{(\vec{r}-\vec{r}_N)}{|\vec{r}-\vec{r}_N|^3}$$

$$= \frac{1}{4\pi\varepsilon_0}\sum_{k=1}^{N}\frac{Q_k(\vec{r}-\vec{r}_k)}{|\vec{r}-\vec{r}_k|^3}$$

➤ 1.6 Lines of Force

Electric field is often visualized by drawing some imaginary field lines which are continuous directed lines drawn so that at any point on a line the direction of the tangent shows the direction of the field. Lines of force have the following properties:

1. They are imaginary lines or curves so that the tangent at any point on the curve gives the direction of the field at that point.



2. They start from positive charge and terminate on negative charge. There is repulsion between the lines.

3. Two lines of force do not intersect with each other, because, if they did, there will be two directions of tangents and hence two direction of electric field at the point of intersection.

4. The lines of force do not form a closed loop. Since formation of a closed loop would imply that some point of the line would simultaneously be a positive charge as well as negative charge.

1.7 Conservative Nature of Electric Field

We can show the conservative nature of electric field by showing that $\vec{\nabla} \times \vec{E} = 0$.
The electric field due to a point charge q at a distance $\vec{r}$ is given by,

$$\vec{E} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r} = \frac{q}{4\pi\varepsilon_0 r^3}\vec{r}$$

$$\vec{\nabla} \times \vec{E} = \frac{q}{4\pi\varepsilon_0}\vec{\nabla} \times \left(\frac{\vec{r}}{r^3}\right) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r^3}(\vec{\nabla} \times \vec{r}) + \vec{\nabla}\left(\frac{1}{r^3}\right) \times \vec{r}\right]$$

$$= 0 + (-3r^{-5}\,\vec{r} \times \vec{r}) = 0 \qquad [\text{since } \vec{\nabla} \times \vec{r} = 0; \vec{r} \times \vec{r} = 0]$$

This shows that $\vec{E}$ is conservative in nature. Alternatively, we can also show that $\oint \vec{E} \cdot d\vec{l} = 0$.

1.8 Electrostatic Potential

Since $\vec{\nabla} \times \vec{E} = 0$, we can write $\vec{E} = -\vec{\nabla}V$. Here V is a scalar function, called electrostatic potential. Now,

$$\vec{E} \cdot d\vec{r} = -\vec{\nabla}V \cdot d\vec{r} = -dV$$

Hence,
$$V(\vec{r}) = -\int \vec{E} \cdot d\vec{r}$$

Since $\vec{E}$ is the electric force on unit charge, we define the potential at a point $\vec{r}$ as the work done in bringing a unit test (positive) charge against the electrostatic force from infinity to that point. We have taken the potential at infinity (reference point) equal to zero.

For a point charge q located at the origin, the potential at a distance r' from the charge is given by

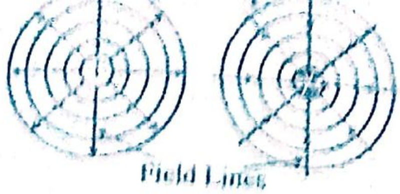
$$V(\vec{r}\,') = -\int_{\infty}^{r} \vec{E}(\vec{r}\,') . d\vec{r}\,' = -\int_{\infty}^{r} \frac{q}{4\pi\varepsilon_0 r'^2}\,dr' = \frac{q}{4\pi\varepsilon_0 r'}$$

If the point charge is located at the position $\vec{r}_0$, the field at $\vec{r}\,'$ is given by,

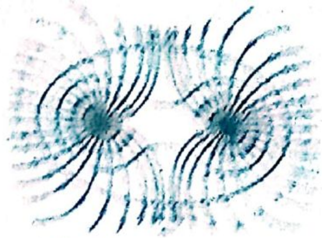
$$V(\vec{r}\,') = \frac{q}{4\pi\varepsilon_0 |\vec{r}\,' - \vec{r}_0|}$$

1.9 Equipotential Surfaces

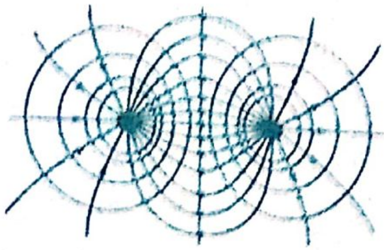
Surface on which electric potential has a constant value is called equipotential surface. For gravity, equipotential surface is the locus of all those points which are at the same height from the surface. For a point charge Q, all points on a spherical surface centered about Q have same potential. Since $\vec{E}$ points in a direction along which potential is decreasing, $\vec{E}$ is normal to the equipotential surface. Some of the equipotential surfaces (closed contours) for different charge configuration are shown below.



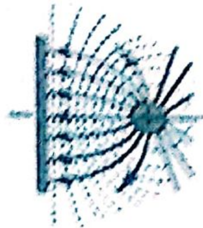
(a) Point charge



(b) Two negative charges



(c) A dipole

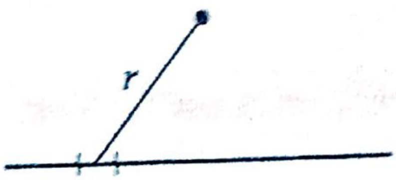


(d) A point charge in front of a charged plane

➤ 1.10 Continuous Charge Distribution

So far, we have considered only point charge, which essentially occupies very small physical space. In practical application, the charges encountered have large number of electronic charges. Hence the charges may be considered continuous at the macroscopic level. The use of a continuous charge density to describe a large number of discrete charges is similar to the use of a continuous mass density to describe air, which actually consists of a large number of discrete molecules.It is possible to have continuous charge distribution along a line, on a surface, or in a volume as illustrated in the Figure.

Line charge : If dq be the small charge in an element of line dl, then the charge per unit length, defined as the line (linear) charge density, and is given by

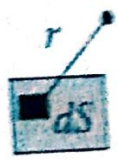


$dq = \lambda dl$

$$\lambda = \frac{dq}{dl}$$

Thus the total charge in the line is, $Q = \int \lambda \, dl$

Surface charge : If dq be the charge in a small surface element dS, then we define the surface charge density as



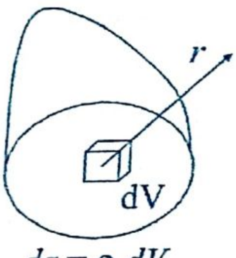
$dq = \sigma dS$

$$\sigma = \frac{dq}{dS}$$

Total charge in the surface is, $Q = \int dq = \int \sigma \, ds$

Volume charge : If dq be the charge in a small volume element dV, then we define the volume charge density as

$$\rho = \frac{dq}{dV}$$



$$dq = \rho\, dV$$

Total charge in the volume is, $Q = \int dq = \int \rho\, dV$

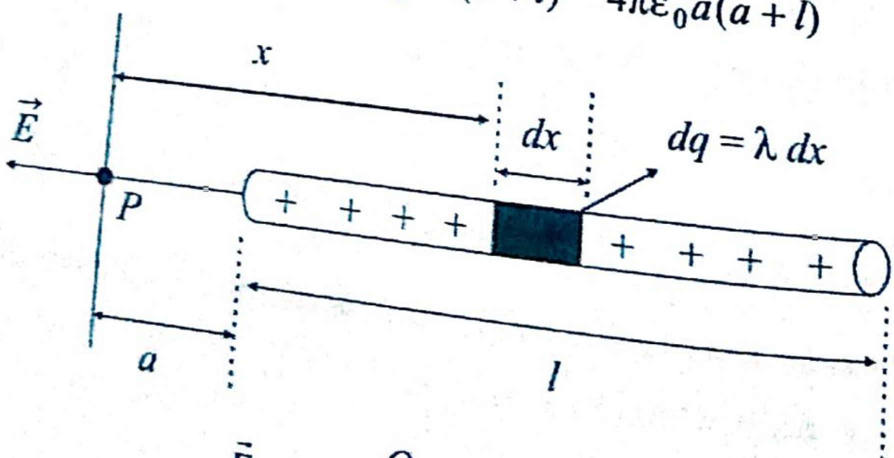
➤ 1.11 Electric Field and Potential for Some Simple Symmetrical Cases

(A) Uniformly Charged Rod (Along the Axis)

Let us consider a rod of length l which is uniformly charged with charge Q and lying along the x-axis. The line charge density of the rod is $\lambda = \frac{Q}{l}$. To find the electric field due to the rod at point P, we consider an element of length dx of the rod at a distance x from P as shown in the figure. The charge $dq = \lambda\, dx$ produces at the point P an electric field, $dE = \frac{dq}{4\pi\varepsilon_0 x^2} = \frac{\lambda\, dx}{4\pi\varepsilon_0 x^2}$. Thus total electric field at P, due to the rod is,

$$E = \int_a^{a+l} \frac{\lambda\, dx}{4\pi\varepsilon_0 x^2} = \frac{\lambda}{4\pi\varepsilon_0}\int_a^{a+l}\frac{dx}{x^2} = \frac{\lambda}{4\pi\varepsilon_0}\left[\frac{1}{a} - \frac{1}{a+l}\right]$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\,\frac{l}{a\,(a+l)} = \frac{Q}{4\pi\varepsilon_0 a(a+l)}$$



Thus,

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 a(a+l)}(-\hat{x})$$

At large distance from the rod, $a >> l$, and the field becomes. $E = \dfrac{Q}{4\pi\varepsilon_0 a^2}$. Thus at large distances, the rod behaves like a point charge.

The potent'al at P can also be calculated in a similar way. The element of charge $dq = \lambda\, dx$ produces at P the potential,

$$dV = \frac{\lambda\, dx}{4\pi\varepsilon_0 x}$$

The total potential at P is therefore,

$$V = \int_a^{a+l} \frac{\lambda\, dx}{4\pi\varepsilon_0 x} = \frac{\lambda}{4\pi\varepsilon_0} \ln\left(\frac{a+l}{a}\right).$$

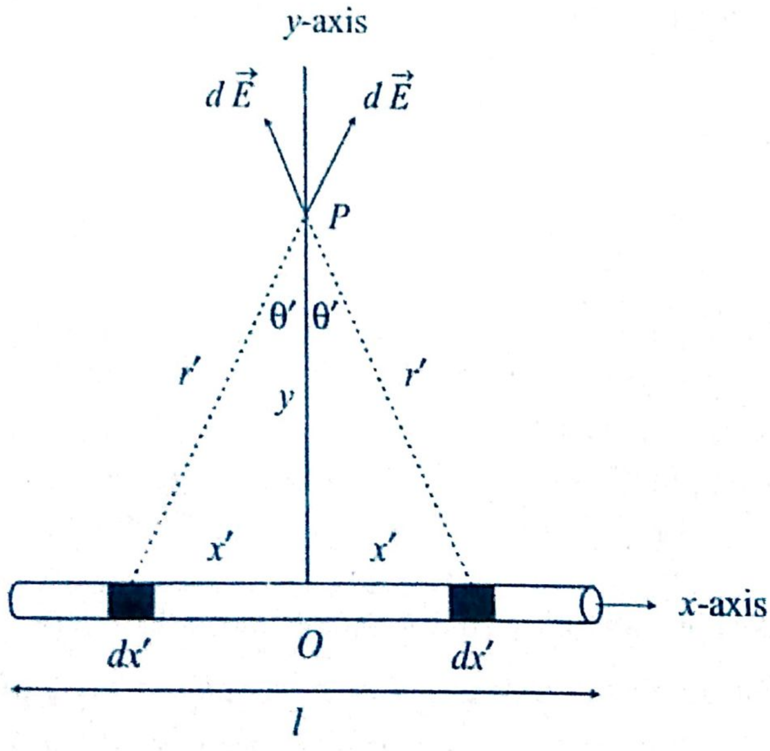
It can be verified from this expression that the electric field is,

$$E = -\frac{dV}{da} = -\frac{\lambda}{4\pi\varepsilon_0}\left(\frac{a}{a+l}\right)\left(-\frac{l}{a^2}\right) = \frac{\lambda}{4\pi\varepsilon_0}\frac{l}{a(a+1)} = \frac{Q}{4\pi\varepsilon_0 a(a+l)}.$$

(B) Uniformly Charged Rod (at the Perpendicular Bisector)

We consider a rod of length L with a uniform charge density λ and a total charge Q is lying along the x-axis, as illustrated in the figure. We wish to compute the electric field at a point P, located at a distance y from the center of the rod along its perpendicular bisector. An element of length dx' carrying charge, $dq = \lambda\, dx'$ contributes to the electric field

$$dE = \frac{1}{4\pi\varepsilon_0}\frac{dq}{(x'^2 + y^2)}$$



Symmetry argument suggests that the x-component of the electric field vanishes and only y-component survives. Now the y-component of dE is

$$dE_y = dE\cos\theta = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\,dx'}{(x'^2+y^2)}\frac{y}{\sqrt{x'^2+y^2}}$$

Thus the total electric field due to the entire rod is

$$E_y = \int dE_y = \frac{\lambda y}{4\pi\varepsilon_0}\int_{-l/2}^{l/2}\frac{dx'}{(x'^2+y^2)^{3/2}} = \frac{1}{4\pi\varepsilon_0}\frac{2\lambda}{y}\frac{l/2}{\sqrt{(l/2)^2+y^2}}$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{1}{y\sqrt{(l/2)^2+y^2}}$$

In the limit where, $y >> l$, the above expression reduces to the "point-charge" limit, *i.e.*,

$$E_y = \frac{q}{4\pi\varepsilon_0 y^2}$$

On the other hand, in infinite length limit $l >> y$, the system has cylindrical symmetry and

$$E_y = \frac{1}{4\pi\varepsilon_0}\frac{2\lambda}{y}.$$

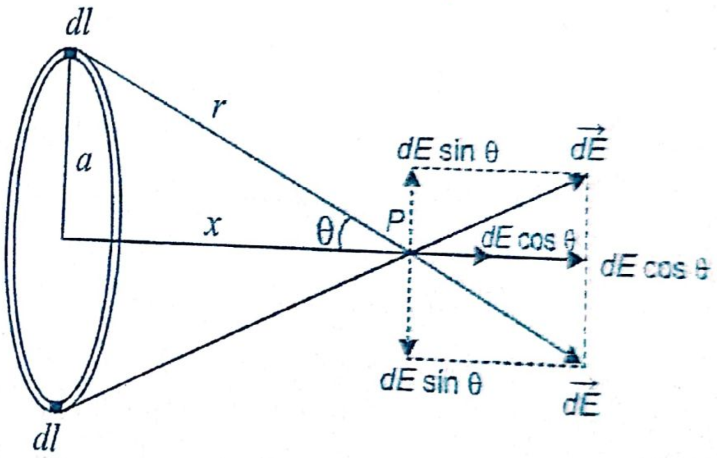
The potential at P can be obtained by performing the integration,

$$V = \frac{1}{4\pi\varepsilon_0}\int_{-l/2}^{l/2}\frac{\lambda\,dx'}{\sqrt{(x'^2+y^2)}}$$

(c) Uniformly Charged Ring

We consider a uniformly charged circular ring of radius a having line charge density λ. The potential at the point P due to an element of length dl of the ring is

$$dV = \frac{\lambda\,dl}{4\pi\varepsilon_0 r} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\,dl}{\sqrt{a^2+x^2}}$$

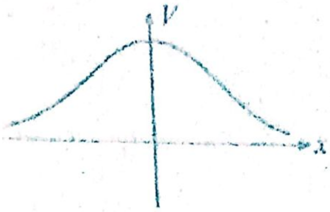


Since contribution to the potential of such small elemental length is same, the net potential at P will be

$$V = \frac{1}{4\pi\varepsilon_0}\int\frac{\lambda\,dl}{\sqrt{a^2+x^2}} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{\sqrt{a^2+x^2}};$$

Q is the total charge on the ring.

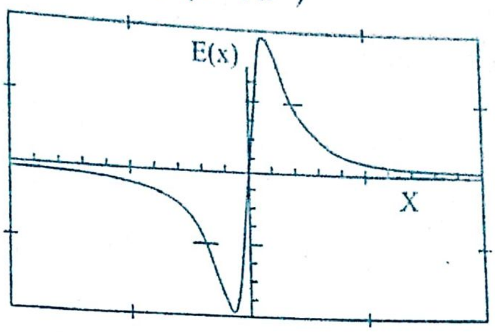
This shows that potential is maximum at $x = 0$ and decreases with increasing x.



The symmetry of the problem suggests that the component of the electric field perpendicular to the axis of the ring will cancel out, while the component along the axis will contribute. Thus the resultant electric field will be along the axis of the ring. The magnitude of the field can be calculated from the expression for V, such that $\vec{E} = -\vec{\nabla} V$.

Thus,
$$\vec{E} = -\vec{\nabla} V = -\frac{dV}{dx}\hat{x}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{Qx}{(a^2+x^2)^{3/2}}\hat{x}; \qquad [\hat{x} = \text{unit vector along } x]$$



The location of maximum electric field is obtained by setting $\frac{dE}{dx} = 0$. It can be shown that E will be maximum at $x = \pm\frac{a}{\sqrt{2}}$.

If the point P is far away from the ring, *i.e.*, when $x >> a$, then, $\vec{E} \cong \frac{1}{4\pi\varepsilon_0}\frac{Q}{x^2}\hat{x}$.

Thus at large distances from the charge distribution, the ring behaves like a point charge. Figure shows the variation of potential (upper) and field (lower) with distance.

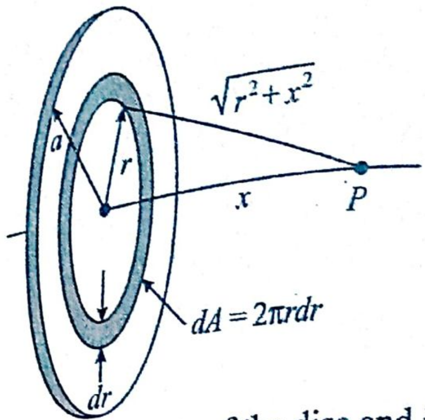
(D) Uniformly Charged Disc

Suppose we have a charge Q smeared out uniformly over a disk of radius a, so that the charge per unit area $\sigma = Q/\pi a^2$. We want to calculate potential V and field E at a distance x from the centre of the disk. We may consider the disk to be made out of an infinite number of infinitesimally thin rings and use above results. One such ring of radius r and thickness dr is shown in the figure. The charge contained on this ring is $(2\pi r\,dr)\,\sigma$. The potential at the point P due to this charge is

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{(2\pi r\,dr)\sigma}{\sqrt{r^2+x^2}} = \frac{\sigma}{2\varepsilon_0}\frac{r\,dr}{2\varepsilon_0\sqrt{r^2+x^2}}$$

The total potential at P, which is the sum of all such contribution from all rings, is given by

$$V = \frac{\sigma}{2\varepsilon_0}\int_0^a \frac{r\,dr}{\sqrt{r^2+x^2}} = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{a^2+x^2} - |x|\right]$$



The potential is maximum at the centre of the disc and is given by,

$$V_{max}=\frac{\sigma a}{2\varepsilon_0}.$$

The electric field at P is given by,

$$\vec{E}=-\vec{\nabla}V=-\frac{dV}{dx}\hat{x}$$

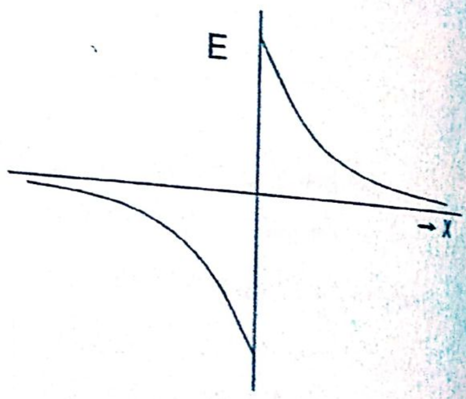
$$=\frac{\sigma}{2\varepsilon_0}\left[1-\frac{x}{\sqrt{a^2+x^2}}\right]\hat{x} \qquad \text{for } x>0$$

and

$$\vec{E}=-\frac{\sigma}{2\varepsilon_0}\left[1-\frac{x}{\sqrt{a^2+x^2}}\right]\hat{x} \qquad \text{for } x<0.$$

At the centre of the disc, $(x=0)$ and $E=\frac{\sigma}{2\varepsilon_0}$.

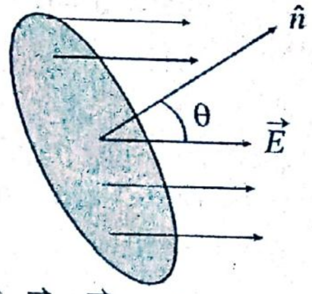
Figure illustrates the variation of field with distance from the centre.



1.12 Electric Flux and Gauss's Law

Calculation of the electric field of a continuous charge distribution can become very complicated for some charge distributions. However, It turns out that, if certain symmetry exists in the charge distribution, it is possible to determine the electric field by means of Gauss's law. To understand Gauss's law we must first understand the concept of electric flux.

The strength of an electric field is proportional to the number of field lines crossing unit area. The number of electric field lines that penetrates a given surface is called an electric flux. Thus, flux is a quantitative measure of the number of lines of a vector field that passes perpendicularly through a surface. Figure shows an electric field $\vec{E}$ passing through a portion of a surface of area A. The area of the surface can be represented by a vector, $\vec{A} = A\hat{n}$, where, $\hat{n}$ is a unit vector normal to the surface.



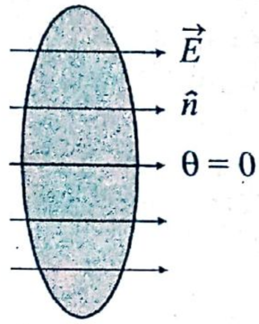
$\vec{E}\,.\,\vec{A} = \vec{E}\,.\,(A\hat{n}) = EA\cos\theta$

The electric flux is thus defined to be,

$$\phi_E = \vec{E} \cdot \vec{A} = EA\cos\theta$$

If $\vec{E}$ points same direction as that of $\hat{n}$, then,

$$\phi_E = \vec{E} \cdot \vec{A} = EA.$$



$\vec{E}\,.\,\vec{A} = \vec{E}\,.\,(A\hat{n}) = EA$

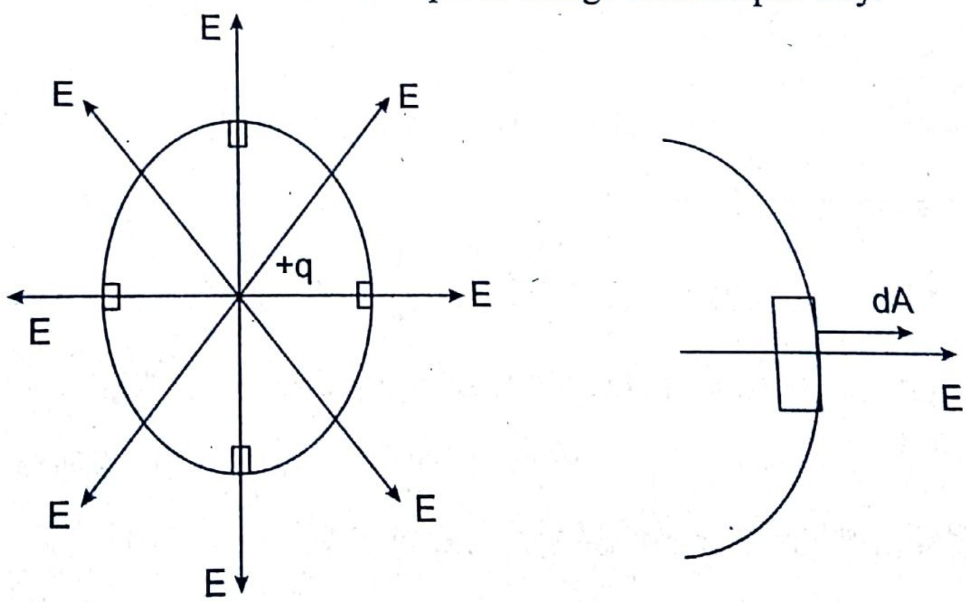
The electric flux is considered positive if the electric field lines are leaving the surface, and negative if the field lines are entering the surface.

Gauss's law

Gauss's law for electrostatics says that the electric flux ϕ_E passing through a closed surface surrounding a point charge q is equal to the total charge q contained within the Gaussian surface divided by the free space permittivity.

Mathematically, $$\phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{q}{\varepsilon_0}$$

We can prove the above formula for a point charge in a simple way.



Let us calculate the electric flux emanating from a positive point charge which is surrounded by an imaginary spherical surface, called a Gaussian surface. The direction of $\vec{E}$ is different at different point. Hence if we consider an infinitesimal surface areas dA, the flux through the infinitesimal area is

$$d\phi_E = \vec{E} \cdot d\vec{A}$$

The total flux out of the Gaussian surface becomes the sum of all the infinitesimal fluxes $d\Phi_E$, through all the infinitesimal areas dA, and is given by

$$\phi_E = \oint d\phi_E = \oint \vec{E} \cdot d\vec{A}$$

Here the integration is performed over the entire closed surface through which the flux is passing. The electric vector $\vec{E}$ is everywhere radial from the point charge q, and $d\vec{A}$ is also everywhere radial, hence

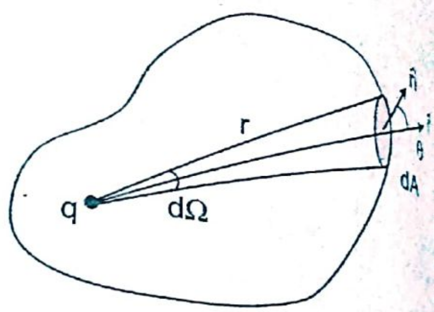
$$\phi_E = \oint \vec{E} \cdot d\vec{A} = \oint \frac{q}{4\pi\varepsilon_0 r^2} dA \qquad [\theta = 0, \cos\theta = 1]$$

Here r is the radius of the spherical Gaussian surface and is a constant. Also, the surface area of a sphere is $4\pi r^2$. Hence,

$$\phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{q}{4\pi\varepsilon_0 r^2}$$

$$\oint dA = \frac{q}{4\pi\varepsilon_0 r^2} 4\pi r^2 = \frac{q}{\varepsilon_0}$$

Although the above equation has been derived for a point charge, it is true, in general, for any kind of charge distribution.



For an arbitrary closed surface enclosing the point charge q the net flux is

$$\phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{q}{4\pi\varepsilon_0} \oint \frac{\hat{r} \cdot d\vec{A}}{r^2}$$

$$= \frac{q}{4\pi\varepsilon_0} \oint \frac{\hat{r} \cdot \hat{n}\, dA}{r^2}$$

Now, $\frac{\hat{r} \cdot \hat{n}\, dA}{r^2} = \frac{dA \cos\theta}{r^2} = d\Omega$ is the solid angle subtended by the surface dA at the location of the point charge q. Now,

$$\phi_E = \frac{q}{4\pi\varepsilon_0} \oint d\Omega = \frac{q}{4\pi\varepsilon_0} 4\pi = \frac{q}{\varepsilon_0}$$

where, $\oint d\Omega = 4\pi$ (steradians) is the total solid angle subtended by the surface at the location of q.

➤ 1.13 Differential Form of Gauss's Law

Now instead of a point charge q, If the charge is distributed over a volume V, with a volume charge density ρ, then the charge q can be written as $q = \oint \rho\, dV$

Hence Gauss's law can be written as,

$$\oint \vec{E} \cdot d\vec{A} = \frac{1}{\varepsilon_0} \int \rho\, dV$$

Using Divergence theorem, left hand side of the above equation can be converted into a volume integral. Thus,

$$\int \vec{\nabla} \cdot \vec{E}\, dV = \frac{1}{\varepsilon_0} \int \rho\, dV$$

Since the volume dV is arbitary, we have,

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

This is the differential form of Gauss's law.

➤ 1.14 Coulomb's Law From Gauss Theorem

We consider a point charge q_0 located somewhere in space. The electric field due to this charge over a spherical surface of radius r surrounding the charge will be uniform and have the same magnitude at all points. Application of Gauss's law on this spherical surface yields,

$$\oint \vec{E} \cdot d\vec{S} = \frac{q_0}{\varepsilon_0}$$

Since $\vec{E}$ and $d\vec{S}$ have the same direction at all points on the surface and $\vec{E}$ is uniform, we can write,

$$E(4\pi r^2) = \frac{q_0}{\varepsilon_0}$$

$$\Rightarrow \qquad E = \frac{q_0}{4\pi\varepsilon_0 r^2}$$

$$\vec{E} = \frac{q_0}{4\pi\varepsilon_0 r^2}\hat{r}$$

If another point charge q_1 is placed at any point on the surface, the force experienced by q_1 will be

$$\vec{F} = q_1\vec{E} = \frac{q_1 q_0}{4\pi\varepsilon_0 r^2}\hat{r}.$$

This is Coulomb's law.

➤ 1.15 Application of Gauss's Law

Gauss's law is very useful in calculating the field due to charge distribution which has some kind of symmetry. A number of cases is discussed below.

1. The Electric Field of a Spherically Symmetric Uniform Charge Distribution (Solid sphere)

Let us consider a spherically symmetric distribution of total electric charge q. We assume that the charge is distributed uniformly over a solid sphere of radius a and has a volume charge density ρ. Hence, we can write,

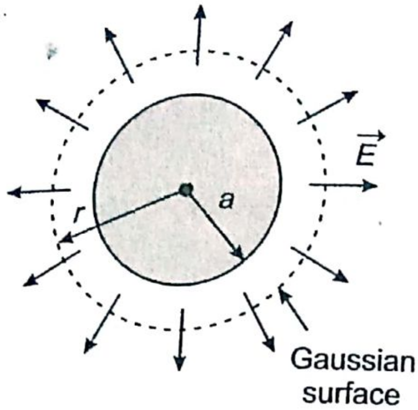
$$q = \frac{4}{3}\pi a^3 \rho.$$

We will find the electric field at points inside and outside the sphere.

(a) The Electric Field Outside the Solid Sphere

To determine the electric field outside the spherical distribution of charge, we draw a spherical Gaussian surface as shown in the figure. The symmetry of problem suggests that the electric field $\vec{E}$ must be radially outward from the distribution. The element of area $d\vec{A}$ of the spherical Gaussian surface is perpend... to the spherical surface and points outward in the radial direction. Hence the ... between the electric field vector $\vec{E}$ and the element of area vector $d\vec{A}$ is $0°$. Applic... of Gauss's law yields,

$$\oint \vec{E} \cdot d\vec{A} = \oint E\, dA = \frac{q}{\varepsilon_0}$$



Since magnitude of $\vec{E}$ is uniform over the Gaussian surface, we have,

$$E \oint dA = \frac{q}{\varepsilon_0}$$

$$E(4\pi r^2) = \frac{q}{\varepsilon_0}$$

$$\Rightarrow \qquad E = \frac{q}{4\pi\varepsilon_0 r^2}$$

Thus outside a spherical charge distribution q, the electric field looks like th... electric field of a point charge.

We can also write, $\quad E = \dfrac{1}{4\pi\varepsilon_0 r^2}\left(\dfrac{4}{3}\pi a^3 \rho\right) = \dfrac{\rho a^3}{3\varepsilon_0 r^2}$

This shows that the electric field E outside the charge distribution is inverse... proportional to the square of the distance r from the center of the charge distributio...

Potential in the region can be calculated from the relation,

$$\vec{E} = -\vec{\nabla}\varphi = -\frac{\partial \varphi}{\partial r}\hat{r}$$

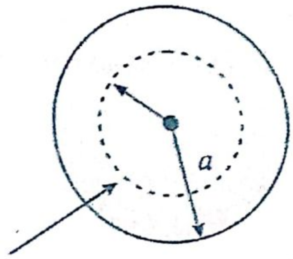
$$\varphi = -\int E\, dr = -\int \frac{\rho a^3}{3\varepsilon_0 r^2} dr = \frac{\rho a^3}{3\varepsilon_0 r} + C \qquad [C = \text{consta...}$$

Since $\varphi \to 0$ as $r \to \infty$, we have $C = 0$, Thus,

$$\varphi = \frac{\rho a^3}{3\varepsilon_0 r}$$

(b) The Electric Field Inside the Solid Sphere

To find the electric field inside the spherical charge distribution, we draw a new spherical Gaussian surface of radius $r\,(r < a)$ that encloses an amount of charge q' as shown in the figure. The Gauss's law in this case gives,



$$\oint \vec{E} \cdot d\vec{A} = \oint E\, dA = \frac{q'}{\varepsilon_0}$$

where, q' is the amount of charge now contained within the new Gaussian surface and is less than the total spherical charge q.

Now the volume charge density ρ is the same for the total charge (q) as it is for the charge enclosed in the new Gaussian surface (q'). That is,

$$\rho = \frac{q}{V} = \frac{q'}{V'}$$

$$\Rightarrow \qquad q' = q\frac{V'}{V} = q\frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi a^3} = q\frac{r^3}{a^3}$$

Hence,

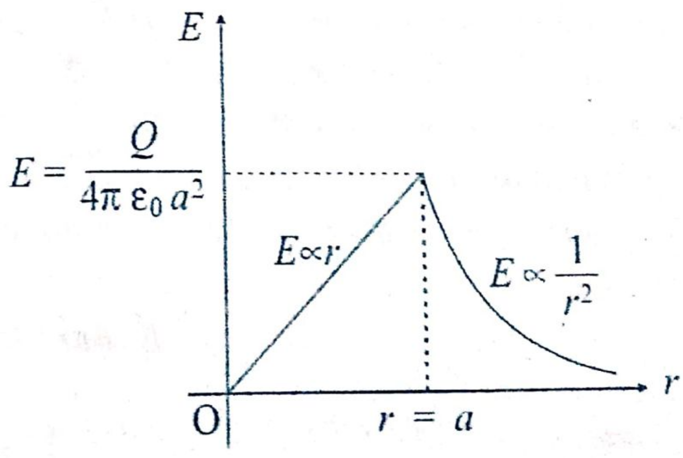
$$\oint E\, dA = \frac{q'}{\varepsilon_0} = \frac{q}{\varepsilon_0}\frac{r^3}{a^3}$$

E being uniform on the Gaussian surface, we can take it out of the integral. Hence, we get,

$$E(4\pi r^2) = \frac{q}{\varepsilon_0}\frac{r^3}{a^3}$$

$$\Rightarrow \qquad E = \frac{q}{4\pi\varepsilon_0}\frac{r}{a^3} = \frac{1}{4\pi\varepsilon_0}\left(\frac{4}{3}\pi a^3 \rho\right)\frac{r}{a^3} = \frac{\rho r}{3\varepsilon_0}.$$

Thus we see that electric field intensity E inside the charge distribution is directly proportional to the radial distance r from the center of the charge distribution.



At the surface of the sphere, the field becomes,

$$E = \frac{\rho a}{3\varepsilon_0}.$$

Figure shows the variation of electric field inside and outside the spherical charge distribution

Here also the potential can be calculated as,

$$\varphi = -\int E\,dr = -\int \frac{\rho r}{3\varepsilon_0} dr = -\frac{\rho r^2}{6\varepsilon_0} + C$$

The boundary condition implies that $\varphi = \frac{\rho a^2}{3\varepsilon_0}$ at $r = a$ (at the surface). Thus,

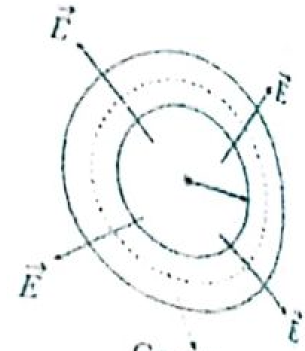
$$\frac{\rho a^2}{3\varepsilon_0} = -\frac{\rho a^2}{6\varepsilon_0} + C$$

$$\Rightarrow \quad C = \frac{\rho a^2}{3\varepsilon_0} + \frac{\rho a^2}{6\varepsilon_0} = \frac{\rho a^2}{2\varepsilon_0}$$

Therefore,
$$\varphi = -\frac{\rho r^2}{6\varepsilon_0} + \frac{\rho a^2}{2\varepsilon_0} = \frac{\rho}{3\varepsilon_0}\left[\frac{3}{2}a^2 - \frac{r^2}{2}\right].$$

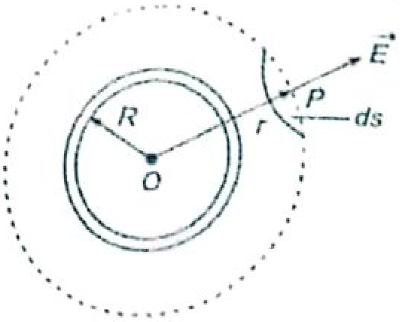
2. Field Inside a Hollow Sphere

As a further example involving spherical symmetry, let us consider a hollow spherically symmetric charge distribution containing a charge q. To find the electric field inside all of charges, we draw a spherical Gaussian surface of radius r, where r is smaller than the inner radius of the hollow ball. By symmetry, the electric field must point in the radial direction and have the same magnitude everywhere on the Gaussian surface. Now Gauss's law gives,



$$\oint \vec{E} \cdot d\vec{A} = E \,.\, 4\pi r^2 = \frac{q}{\varepsilon_0}$$

But the Gaussian surface encloses no charge at all ($q = 0$). Therefore,



$$E \,.\, 4\pi r^2 = \frac{q}{\varepsilon_0} = 0$$

$$\Rightarrow \quad E = 0.$$

Thus there is no field inside the hollow sphere of charge.

Field Outside the Hollow Sphere

To calculate the field outside the hollow sphere, we consider a spherical Gaussian surface of radius r, greater than the outer radius of the hollow sphere. By symmetry, field will be directed radially outwards and has same magnitude at all points on this Gaussian surface. Hence, Gauss's law gives,

$$\oint \vec{E} \cdot d\vec{A} = E \,.\, 4\pi r^2 = \frac{q}{\varepsilon_0}$$

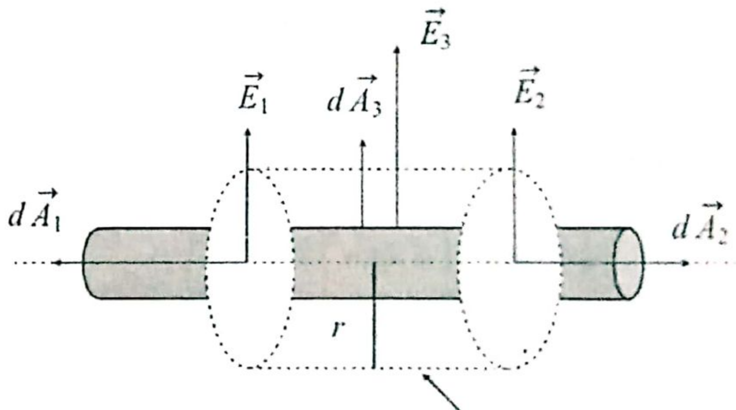
$$\Rightarrow \qquad E = \frac{q}{4\pi\varepsilon_0 r^2}$$

$$\vec{E} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r}.$$

Thus the hollow sphere behaves like a point charge at outside point.

3. Infinitely Long Rod of Uniform Charge Density

Let us consider an infinitely long rod of negligible radius having a uniform line charge density λ. We wish to calculate the electric field at a distance r from the wire. The geometry of the problem suggests that it has cylindrical symmetry and the electric field $\vec{E}$ must point radially away from the axis of the rod. The magnitude of the electric field is constant on cylindrical surfaces of radius r. Therefore, we choose a coaxial cylinder as our Gaussian surface.



The amount of charge enclosed by the cylindrical Gaussian surface of radius r and length l is $q = \lambda l$. The Gaussian surface consists of three parts: two flat ends and the curved side wall. Now the contribution to the flux through the two end faces is zero, as direction of $\vec{E}$ is normal to the direction of the surface $d\vec{A}$. So only contribution to the flux comes from the curved surface. Application of Gauss's law therefore gives,

$$\oint \vec{E} \cdot d\vec{A} = \oint E \, dA = \frac{q}{\varepsilon_0} = \frac{\lambda l}{\varepsilon_0}$$

As $|\vec{E}|$ is same throughout the curved Gaussian surface, we have,

$$E\,(2\pi r l) = \frac{\lambda l}{\varepsilon_0}$$

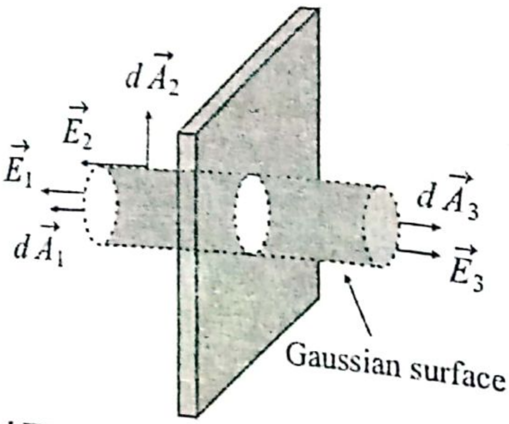
$$\Rightarrow \quad E = \frac{\lambda}{2\pi r \varepsilon_0}$$

It is important to note here that the value of the electric field does not depend on the length l of the cylinder, and depends only on the distance r from the axis.

4. The Electric Field Due to an Infinite Charged Plane

An infinite sheet of charge possesses a planar symmetry. The electric field $\vec{E}$ must point perpendicularly away from the plane, $\vec{E} = E\hat{k}$. We assume that the charge q is uniformly distributed over the sheet and has a surface charge density, $\sigma = q/A$. To determine the electric field at a distance r from the sheet, we first draw a cylindrical Gaussian surface as shown in the figure. As before, we can break the entire surface of the cylinder into three surfaces: two flat ends and the curved side wall. Applying Gauss's law to this surface, we get

$$\oint \vec{E} \cdot d\vec{A} = \frac{q}{\varepsilon_0}$$



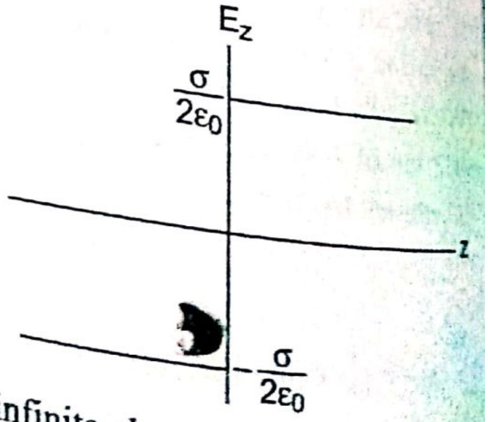
For the surface I and III, the field is parallel to the direction of the surface, while for surface II, the field is normal to the surface. Hence surface II contributes nothing to the flux. Thus,

$$\oint \vec{E} \cdot d\vec{A} = \int E\, dA + \int E\, dA = \frac{q}{\varepsilon_0}$$

$$EA + EA = \frac{q}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}$$

$$\Rightarrow \quad E = \frac{\sigma}{2\varepsilon_0} \qquad [A = \text{Area of the end cap}]$$

The result shows that the electric field is independent of the distance from the sheet of charge, which is quite astonishing. This means the field is same at very small distance from the sheet as well as at very large distance from the infinite plane. This result of the infinite sheet of charge can be used as a good approximation if the electric field is found very close to the finite sheet of charge. At a very close distance, the finite sheet of charge can be approximated as an infinite sheet of charge. In vector notation we can write,



$$\vec{E} = \begin{cases} \frac{\sigma}{2\varepsilon_0}\hat{k}, & z>0 \\ -\frac{\sigma}{2\varepsilon_0}\hat{k}, & z<0 \end{cases}$$

Thus, we see that the electric field due to an infinite large plane is uniform in space, the result of which is plotted in the figure.

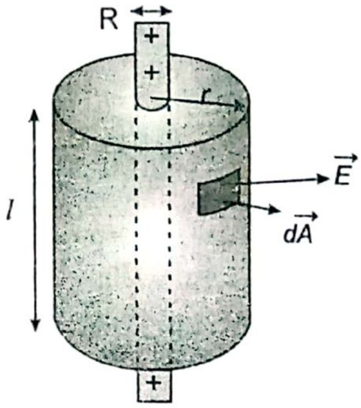
5. Electric Field Due to Cylindrically Symmetric Charge Distribution

For a **cylindrically symmetric** charge distribution, the charge density depends only upon the distance from the axis of the cylinder and must not vary along the axis of the cylinder. Let us consider a solid cylinder of radius R and length l charged uniformly with a total charge q. The volume charge density of the distribution is

$$\rho = \frac{q}{\pi R^2 l}$$

Field Outside the Cylinder ($r > R$) :

To *find the electric field at a distance* $r > R$ from the axis, we consider a coaxial cylindrical Gaussian surface of length l and radius r. By symmetry, the electric field will be directed along the radius of the Gaussian cylinder. The two end faces of the Gaussian surface will contribute nothing to the flux and only contribution comes from the curved surface. Hence, application of Gauss's law gives,



$$\oint \vec{E}\cdot d\vec{A} = \oint E\, dA = \frac{q}{\varepsilon_0}$$

$$E\,(2\pi r l) = \frac{q}{\varepsilon_0}$$

$$\Rightarrow \qquad E = \frac{q}{2\pi\varepsilon_0 r l} = \frac{(\pi R^2 l)\rho}{2\pi\varepsilon_0 r l} = \frac{R^2\rho}{2\varepsilon_0 r}$$

$$\vec{E} = \frac{R^2\rho}{2\varepsilon_0 r}\hat{r}$$

Field Inside the Cylinder ($r < R$) :

This time we consider the Gaussian cylinder of radius r ($r < R$) inside the charged cylinder of radius R. *Symmetry arguments remains same as before. Thus Gauss's law gives,*

$$\oint \vec{E}\cdot d\vec{A} = \oint E\, dA = \frac{q'}{\varepsilon_0};$$

Here q' = charge enclosed by Gaussian cylinder.

$$E\,(2\pi r l) = \frac{q'}{\varepsilon_0} \Rightarrow E = \frac{q'}{2\pi\varepsilon_0 r l} = \frac{(\pi r^2 l)\rho}{2\pi\varepsilon_0 r l} = \frac{\rho r}{2\varepsilon_0}$$