2

Fourier Transforms and Their Applications

"The profound study of nature is the most fertile source of mathematical discoveries."

Joseph Fourier

"The theory of Fourier series and integrals has always had major difficulties and necessitated a large mathematical apparatus in dealing with questions of convergence. It engendered the development of methods of summation, although these did not lead to a completely satisfactory solution of the problem. ... For the Fourier transform, the introduction of distributions (hence, the space **S**) is inevitable either in an explicit or hidden form. ... As a result one may obtain all that is desired from the point of view of the continuity and inversion of the Fourier transform."

Laurent Schwartz

2.1 Introduction

Many linear boundary value and initial value problems in applied mathematics, mathematical physics, and engineering science can be effectively solved by the use of the Fourier transform, the Fourier cosine transform, or the Fourier sine transform. These transforms are very useful for solving differential or integral equations for the following reasons. First, these equations are replaced by simple algebraic equations, which enable us to find the solution of the transform function. The solution of the given equation is then obtained in the original variables by inverting the transform solution. Second, the Fourier transform of the elementary source term is used for determination of the fundamental solution that illustrates the basic ideas behind the construction and implementation of Green's functions. Third, the transform solution combined with the convolution theorem provides an elegant representation of the solution for the boundary value and initial value problems.

We begin this chapter with a formal derivation of the Fourier integral formulas. These results are then used to define the Fourier, Fourier cosine, and Fourier sine transforms. This is followed by a detailed discussion of the basic operational properties of these transforms with examples. Special attention is given to convolution and its main properties. Sections 2.10 and 2.11 deal with applications of the Fourier transform to the solution of ordinary differential equations and integral equations. In Section 2.12, a wide variety of partial differential equations are solved by the use of the Fourier transform method. The technique that is developed in this and other sections can be applied with little or no modification to different kinds of initial and boundary value problems that are encountered in applications. The Fourier cosine and sine transforms are introduced in Section 2.13. The properties and applications of these transforms are discussed in Sections 2.14 and 2.15. This is followed by evaluation of definite integrals with the aid of Fourier transforms. Section 2.17 is devoted to applications of Fourier transforms in mathematical statistics. The multiple Fourier transforms and their applications are discussed in Section 2.18.

2.2 The Fourier Integral Formulas

A function $f(x)$ is said to satisfy *Dirichlet's conditions* in the interval $-a < x < a$, if

(i) $f(x)$ has only a finite number of finite discontinuities in $-a < x < a$ and has no infinite discontinuities.

(ii) $f(x)$ has only a finite number of maxima and minima in $-a < x < a$.

From the theory of Fourier series we know that if $f(x)$ satisfies the Dirichlet conditions in $-a < x < a$, it can be represented as the complex Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} a_n \exp(in\pi x/a), \tag{2.2.1}$$

where the coefficients are

$$a_n = \frac{1}{2a}\int_{-a}^{a} f(\xi)\exp(-in\pi\xi/a)d\xi. \tag{2.2.2}$$

This representation is evidently periodic of period $2a$ in the interval. However, the right-hand side of (2.2.1) cannot represent $f(x)$ *outside* the interval $-a < x < a$ unless $f(x)$ is periodic of period $2a$. Thus, problems on finite intervals lead to Fourier series, and problems on the whole line $-\infty < x < \infty$ lead to the

Fourier integrals. We now attempt to find an integral representation of a non-periodic function $f(x)$ in $(-\infty, \infty)$ by letting $a \to \infty$. As the interval grows $(a \to \infty)$ the values $k_n = \frac{n\pi}{a}$ become closer together and form a dense set. If we write $\delta k_n = (k_{n+1} - k_n) = \frac{\pi}{a}$ and substitute coefficients a_n into (2.2.1), we obtain

$$f(x) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} (\delta k_n) \left[\int_{-a}^{a} f(\xi) \exp(-i\xi k_n) d\xi \right] \exp(ixk_n). \tag{2.2.3}$$

In the limit as $a \to \infty$, k_n becomes a continuous variable k and δk_n becomes dk. Consequently, the sum can be replaced by the integral in the limit and (2.2.3) reduces to the result

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\xi) e^{-ik\xi} d\xi \right] e^{ikx} dk. \tag{2.2.4}$$

This is known as the celebrated *Fourier integral formula* . Although the above arguments do not constitute a rigorous proof of (2.2.4), the formula is correct and valid for functions that are piecewise continuously differentiable in every finite interval and is absolutely integrable on the whole real line.

A function $f(x)$ is said to be *absolutely integrable* on $(-\infty, \infty)$ if

$$\int_{-\infty}^{\infty} |f(x)| dx < \infty \tag{2.2.5}$$

exists.

It can be shown that the formula (2.2.4) is valid under more general conditions. The result is contained in the following theorem:

THEOREM 2.2.1 *(The Fourier Integral Theorem).*
If $f(x)$ satisfies Dirichlet's conditions in $(-\infty, \infty)$, and is absolutely integrable on $(-\infty, \infty)$, then the Fourier integral (2.2.4) converges to the function $\frac{1}{2}[f(x+0) + f(x-0)]$ at a finite discontinuity at x. In other words,

$$\frac{1}{2}[f(x+0) + f(x-0)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \left[\int_{-\infty}^{\infty} f(\xi) e^{-ik\xi} d\xi \right] dk. \tag{2.2.6}$$

This is usually called the *Fourier integral theorem.*

If the function $f(x)$ is continuous at point x, then $f(x+0) = f(x-0) = f(x)$, then (2.2.6) reduces to (2.2.4).

The Fourier integral theorem was originally stated in Fourier's famous treatise entitled *La Théorie Analytique da la Chaleur* (1822), and its deep significance was recognized by mathematicians and mathematical physicists. Indeed, this theorem is one of the most monumental results of modern mathematical analysis and has widespread physical and engineering applications.

We express the exponential factor $\exp[ik(x-\xi)]$ in (2.2.4) in terms of trigonometric functions and use the even and odd nature of the cosine and the sine functions respectively as functions of k so that (2.2.4) can be written as

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty} dk \int_{-\infty}^{\infty} f(\xi)\cos k(x-\xi)d\xi. \tag{2.2.7}$$

This is another version of the *Fourier integral formula*. In many physical problems, the function $f(x)$ vanishes very rapidly as $|x|\to\infty$, which ensures the existence of the repeated integrals as expressed.

We now assume that $f(x)$ is an even function and expand the cosine function in (2.2.7) to obtain

$$f(x) = f(-x) = \frac{2}{\pi}\int_{0}^{\infty} \cos kx\, dk \int_{0}^{\infty} f(\xi)\cos k\xi\, d\xi. \tag{2.2.8}$$

This is called the *Fourier cosine integral formula*.

Similarly, for an odd function $f(x)$, we obtain the *Fourier sine integral formula*

$$f(x) = -f(-x) = \frac{2}{\pi}\int_{0}^{\infty} \sin\, kx\, dk \int_{0}^{\infty} f(\xi)\sin k\xi\, d\xi. \tag{2.2.9}$$

These integral formulas were discovered independently by Cauchy in his work on the propagation of waves on the surface of water.

2.3 Definition of the Fourier Transform and Examples

We use the Fourier integral formula (2.2.4) to give a formal definition of the Fourier transform.

DEFINITION 2.3.1 *The Fourier transform of $f(x)$ is denoted by $\mathscr{F}\{f(x)\} = F(k)$, $k \in \mathbb{R}$, and defined by the integral*

$$\mathscr{F}\{f(x)\} = F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} f(x)dx, \tag{2.3.1}$$

where $\mathscr{F}$ is called the Fourier transform operator or the Fourier transformation and the factor $\frac{1}{\sqrt{2\pi}}$ is obtained by splitting the factor $\frac{1}{2\pi}$ involved in

(2.2.4). This is often called the complex Fourier transform. A sufficient condition for $f(x)$ to have a Fourier transform is that $f(x)$ is absolutely integrable on $(-\infty, \infty)$. The convergence of the integral (2.3.1) follows at once from the fact that $f(x)$ is absolutely integrable. In fact, the integral converges uniformly with respect to k. Physically, the Fourier transform $F(k)$ can be interpreted as an integral superposition of an infinite number of sinusoidal oscillations with different wavenumbers k (or different wavelengths $\lambda = \frac{2\pi}{k}$).

Thus, the definition of the Fourier transform is restricted to absolutely integrable functions. This restriction is too strong for many physical applications. Many simple and common functions, such as constant function, trigonometric functions $\sin ax$, $\cos ax$, exponential functions, and $x^n H(x)$ do not have Fourier transforms, even though they occur frequently in applications. The integral in (2.3.1) fails to converge when $f(x)$ is one of the above elementary functions. This is a very unsatisfactory feature of the theory of Fourier transforms. However, this unsatisfactory feature can be resolved by means of a natural extension of the definition of the Fourier transform of a generalized function, $f(x)$ in (2.3.1). We follow Lighthill (1958) and Jones (1982) to discuss briefly the theory of the Fourier transforms of good functions.

The inverse Fourier transform, denoted by $\mathscr{F}^{-1}\{F(k)\} = f(x)$, is defined by

$$\mathscr{F}^{-1}\{F(k)\} = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx}\, F(k)\, dk, \tag{2.3.2}$$

where $\mathscr{F}^{-1}$ is called the inverse Fourier transform operator.

Clearly, both $\mathscr{F}$ and $\mathscr{F}^{-1}$ are linear integral operators. In applied mathematics, x usually represents a space variable and $k(= \frac{2\pi}{\lambda})$ is a wavenumber variable where λ is the wavelength. However, in electrical engineering, x is replaced by the time variable t and k is replaced by the frequency variable $\omega(= 2\pi\nu)$ where ν is the frequency in cycles per second. The function $F(\omega) = \mathscr{F}\{f(t)\}$ is called the *spectrum* of the *time signal function* $f(t)$. In electrical engineering literature, the Fourier transform pairs are defined slightly differently by

$$\mathscr{F}\{f(t)\} = F(\nu) = \int_{-\infty}^{\infty} f(t) e^{-2\pi\nu i t} dt, \tag{2.3.3}$$

and

$$\mathscr{F}^{-1}\{F(\nu)\} = f(t) = \int_{-\infty}^{\infty} F(\nu) e^{2\pi i \nu t} d\nu = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega, \tag{2.3.4}$$

where $\omega = 2\pi\nu$ is called the *angular frequency*. The Fourier integral formula implies that any function of time $f(t)$ that has a Fourier transform can be equally specified by its spectrum. Physically, the signal $f(t)$ is represented as an integral superposition of an infinite number of sinusoidal oscillations with different frequencies ω and complex amplitudes $\frac{1}{2\pi}F(\omega)$. Equation (2.3.4) is called the *spectral resolution* of the signal $f(t)$, and $\frac{F(\omega)}{2\pi}$ is called the *spectral density.* In summary, the Fourier transform maps a function (or signal) of time t to a function of frequency ω. In the same way as the Fourier series expansion of a periodic function decomposes the function into harmonic components, the Fourier transform generates a function (or signal) of a continuous variable whose value represents the frequency content of the original signal. This led to the successful use of the Fourier transform to analyze the form of time-varying signals in electrical engineering and seismology.

Next we give examples of Fourier transforms.

Example 2.3.1 Find the Fourier transform of $\exp(-ax^2)$. Then find $\int_{-\infty}^{\infty} x^2\, e^{-ax^2} dx$.

In fact, we prove

$$F(k) = \mathscr{F}\{\exp(-ax^2)\} = \frac{1}{\sqrt{2a}} \exp\left(-\frac{k^2}{4a}\right), \quad a > 0. \tag{2.3.5}$$

Here we have, by definition,

$$\begin{aligned} F(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx - ax^2} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-a\left(x + \frac{ik}{2a}\right)^2 - \frac{k^2}{4a}\right] dx \\ &= \frac{1}{\sqrt{2\pi}} \exp(-k^2/4a) \int_{-\infty}^{\infty} e^{-ay^2} dy = \frac{1}{\sqrt{2a}} \exp\left(-\frac{k^2}{4a}\right), \end{aligned}$$

in which the change of variable $y = x + \frac{ik}{2a}$ is used. The above result is correct, but the change of variable can be justified by the method of complex analysis because $(ik/2a)$ is complex. If $a = \frac{1}{2}$

$$\mathscr{F}\{e^{-x^2/2}\} = e^{-k^2/2}. \tag{2.3.6}$$

This shows $\mathscr{F}\{f(x)\} = f(k)$. Such a function is said to be *self-reciprocal* under the Fourier transformation. Graphs of $f(x) = \exp(-ax^2)$ and its Fourier transform is shown in Figure 2.1 for $a = 1$.

Alternatively, (2.3.5) can be proved as follows:

$$F'(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (-ix) e^{-ikx-ax^2} dx$$

$$= \frac{i}{2a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[(-2ax) e^{-ax^2} \right] e^{-ikx} dx$$

which is, integrating by parts,

$$F'(k) = \frac{i}{2a\sqrt{2\pi}} \left\{ \left[e^{-ax^2-ikx} \right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} (ik) e^{-ax^2-ikx}\, dx \right\}$$

$$= -\frac{k}{2a} F(k).$$

The solution for $F(k)$ is $F(k) = A(k) e^{-\frac{k^2}{4a}}$ so that

$$A(0) = F(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ax^2} dx = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{\pi}{a}} = \frac{1}{\sqrt{2a}}.$$

Thus, $F(k) = \frac{1}{\sqrt{2a}}\, e^{-\frac{k^2}{4a}}$.

Using (2.3.5), we prove that

$$I = \int_{-\infty}^{\infty} e^{-ax^2} = \sqrt{\frac{\pi}{a}}, \quad a > 0.$$

It follows from (2.3.5) that

$$\int_{-\infty}^{\infty} e^{-ikx-ax^2} dx = \sqrt{2\pi}\, F(k) = \sqrt{\frac{\pi}{a}}\, e^{-\frac{k^2}{4a}}.$$

This is true for all k, and hence, putting $k=0$ we obtain the desired result. Differentiating once under the integral sign with respect to a gives

$$\int_{-\infty}^{\infty} x^2\, e^{-ax^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{a^3}} = \frac{1}{2a} \sqrt{\frac{\pi}{a}}.$$

□

Differentiating the integral I, n times with respect to a, yields

$$\int_{-\infty}^{\infty} x^{2n}\, e^{-ax^2} dx = \frac{1.3.5......(2n-1)}{2^n} \sqrt{\frac{\pi}{a^{2n+1}}}$$

$$= \frac{1.3.5......(2n-1)}{(2a)^n} \sqrt{\frac{\pi}{a}}.$$

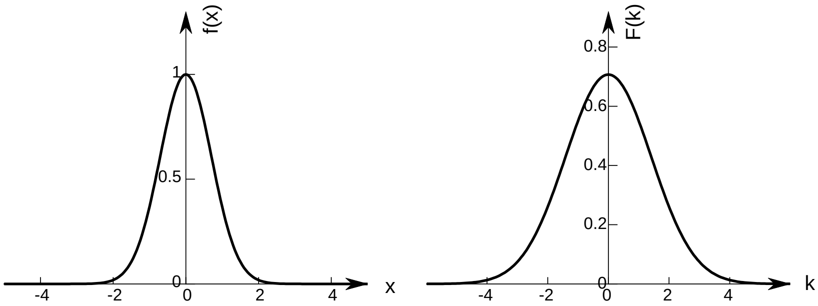

Figure 2.1 Graphs of $f(x)=\exp(-ax^2)$ and $F(k)$ with $a=1$.

Example 2.3.2 Find the Fourier transform of $\exp(-a|x|)$, i.e.,

$$\mathscr{F}\{\exp(-a|x|)\} = \sqrt{\frac{2}{\pi}} \cdot \frac{a}{(a^2+k^2)}, \quad a>0. \tag{2.3.7}$$

Here we can write

$$\begin{aligned}\mathscr{F}\left\{e^{-a|x|}\right\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|-ikx}dx \\ &= \frac{1}{\sqrt{2\pi}} \left[\int_0^{\infty} e^{-(a+ik)x}dx + \int_{-\infty}^{0} e^{(a-ik)x}dx\right] \\ &= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{a+ik} + \frac{1}{a-ik}\right] = \sqrt{\frac{2}{\pi}} \frac{a}{(a^2+k^2)}.\end{aligned}$$

We note that $f(x)=\exp(-a|x|)$ decreases rapidly at infinity, it is not differentiable at $x=0$. Graphs of $f(x) = \exp(-a|x|)$ and its Fourier transform is displayed in Figure 2.2 for $a=1$. $\square$

Example 2.3.3 Find the Fourier transform of

$$f(x) = \left(1-\frac{|x|}{a}\right) H\left(1-\frac{|x|}{a}\right),$$

where $H(x)$ is the *Heaviside unit step function* defined by

$$H(x) = \left\{\begin{array}{ll} 1, & x>0 \\ 0, & x<0 \end{array}\right\}. \tag{2.3.8}$$

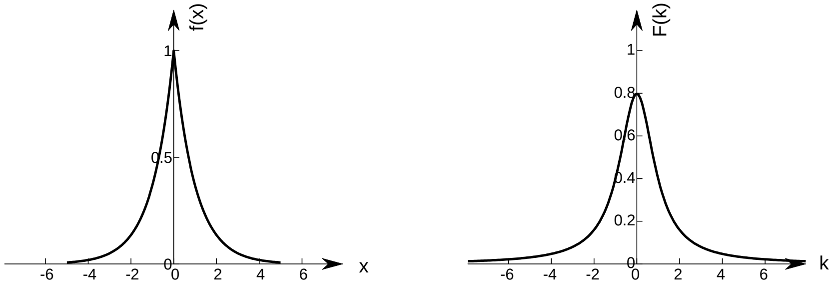

Figure 2.2 Graphs of $f(x) = \exp(-a|x|)$ and $F(k)$ with $a = 1$.

Or, more generally,

$$H(x-a) = \begin{Bmatrix} 1, & x > a \\ 0, & x < a \end{Bmatrix}, \tag{2.3.9}$$

where a is a fixed real number. So the Heaviside function $H(x-a)$ has a finite discontinuity at $x = a$.

$$\begin{aligned}
\mathscr{F}\{f(x)\} &= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-ikx} \left(1 - \frac{|x|}{a}\right) dx = \frac{2}{\sqrt{2\pi}} \int_{0}^{a} \left(1 - \frac{x}{a}\right) \cos kx \, dx \\
&= \frac{2a}{\sqrt{2\pi}} \int_{0}^{1} (1-x) \cos(akx) dx = \frac{2a}{\sqrt{2\pi}} \int_{0}^{1} (1-x) \frac{d}{dx} \left(\frac{\sin akx}{ak}\right) dx \\
&= \frac{2a}{\sqrt{2\pi}} \int_{0}^{1} \frac{\sin(akx)}{ak} dx = \frac{a}{\sqrt{2\pi}} \int_{0}^{1} \frac{d}{dx} \left[\frac{\sin^2\left(\frac{akx}{2}\right)}{\left(\frac{ak}{2}\right)^2} \right] dx \\
&= \frac{a}{\sqrt{2\pi}} \, \frac{\sin^2\left(\frac{ak}{2}\right)}{\left(\frac{ak}{2}\right)^2}.
\end{aligned} \tag{2.3.10}$$

□

Example 2.3.4 Find the Fourier transform of the characteristic function $\chi_{[-a,a]}(x)$, where

$$\chi_{[-a,a]}(x) = H(a - |x|) = \begin{Bmatrix} 1, & |x| < a \\ 0, & |x| > a \end{Bmatrix}. \tag{2.3.11}$$

We have

$$F_a(k) = \mathscr{F}\{\chi_{[-a,a]}(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \chi_{[-a,a]}(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-ikx}\, dx = \sqrt{\frac{2}{\pi}} \left(\frac{\sin ak}{k}\right). \quad (2.3.12)$$

Graphs of $f(x) = \chi_{[-a,a]}(x)$ and its Fourier transform are shown in Figure 2.3 for $a = 1$.

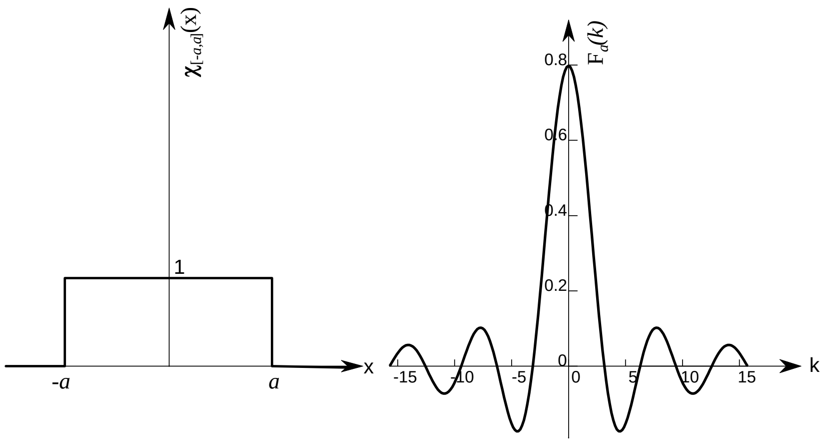

Figure 2.3 Graphs of $\chi_{[-a,a]}(x)$ and $F_a(k)$ with $a = 1$.

□

2.4 Fourier Transforms of Generalized Functions

The natural way to define the Fourier transform of a generalized function, is to treat $f(x)$ in (2.3.1) as a generalized function. The advantage of this is that every generalized function has a Fourier transform and an inverse Fourier transform, and that the ordinary functions whose Fourier transforms are of interest form a subset of the generalized functions. We would not go into great detail, but refer to the famous books of Lighthill (1958) and Jones (1982) for the introduction to the subject of generalized functions.

A *good function*, $g(x)$ is a function in $C^{\infty}(\mathbb{R})$ that decays sufficiently rapidly that $g(x)$ and all of its derivatives decay to zero faster than $|x|^{-N}$ as $|x| \to \infty$ for all $N > 0$.

DEFINITION 2.4.1 *Suppose a real or complex valued function $g(x)$ is defined for all $x \in \mathbb{R}$ and is infinitely differentiable everywhere, and suppose that each derivative tends to zero as $|x| \to \infty$ faster that any positive power of $\left(x^{-1}\right)$, or in other words, suppose that for each positive integer N and n,*

$$\lim_{|x| \to \infty} x^N g^{(n)}(x) = 0,$$

then $g(x)$ is called a good function.

Usually, the class of good functions is represented by $\mathbb{S}$. The good functions play an important role in Fourier analysis because the inversion, convolution, and differentiation theorems as well as many others take simple forms with no problem of convergence. The rapid decay and infinite differentiability properties of good functions lead to the fact that the Fourier transform of a good function is also a good function.

Good functions also play an important role in the theory of generalized functions. A good function of bounded support is a special type of good function that also plays an important part in the theory of generalized functions. Good functions also have the following important properties. The sum (or difference) of two good functions is also a good function. The product and convolution of two good functions are good functions. The derivative of a good function is a good function; $x^n g(x)$ is a good function for all non-negative integers n whenever $g(x)$ is a good function. A good function belongs to L^p (a class of p^{th} power Lebesgue integrable functions) for every p in $1 \le p \le \infty$. The integral of a good function is not necessarily good. However, if $\phi(x)$ is a good function, then the function g defined for all x by

$$g(x) = \int_{-\infty}^{x} \phi(t)\, dt$$

is a good function if and only if $\int_{-\infty}^{\infty} \phi(t)\, dt$ exists.

Good functions are not only continuous, but are also uniformly continuous in $\mathbb{R}$ and absolutely continuous in $\mathbb{R}$. However, a good function cannot be necessarily represented by a Taylor series expansion in every interval. As an example, consider a good function of bounded support

$$g(x) = \left\{ \begin{array}{rl} \exp[-(1-x^2)^{-1}], & \text{if } |x| < 1 \\ 0, & \text{if } |x| \ge 1 \end{array} \right\}.$$

The function g is infinitely differentiable at $x = \pm 1$, as it must be in order to be good. It does not have a Taylor series expansion in every interval, because a Taylor expansion based on the various derivatives of g for any point having $|x| > 1$ would lead to zero value for all x.

For example, $\exp(-x^2)$, $x\,\exp(-x^2)$, $\left(1+x^2\right)^{-1}\exp(-x^2)$, and $\operatorname{sech}^2 x$ are good functions, while $\exp(-|x|)$ is not differentiable at $x=0$, and the function $\left(1+x^2\right)^{-1}$ is not a good function as it decays too slowly as $|x| \to \infty$.

A sequence of good functions, $\{f_n(x)\}$ is called *regular* if, for any good function $g(x)$,

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\, g(x)\, dx \tag{2.4.1}$$

exists. For example, $f_n(x) = \frac{1}{n}\,\phi(x)$ is a regular sequence for any good function $\phi(x)$, if

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\, g(x)\, dx = \lim_{n\to\infty} \frac{1}{n} \int_{-\infty}^{\infty} \phi(x)\, g(x)\, dx = 0\,.$$

Two regular sequences of good functions are equivalent if, for any good function $g(x)$, the limit (2.4.1) exists and is the same for each sequence.

A *generalized function*, $f(x)$, is a regular sequence of good functions, and two generalized functions are equal if their defining sequences are equivalent. Generalized functions are, therefore, only defined in terms of their action on integrals of good functions if

$$\langle f,\, g\rangle = \int_{-\infty}^{\infty} f(x)\, g(x)\, dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\, g(x)\, dx = \lim_{n\to\infty} \langle f_n,\, g\rangle \tag{2.4.2}$$

for any good function, $g(x)$, where the symbol $\langle f,\, g\rangle$ is used to denote the action of the generalized function $f(x)$ on the good function $g(x)$, or $\langle f,\, g\rangle$ represents the number that f associates with g. If $f(x)$ is an ordinary function such that $\left(1+x^2\right)^{-N} f(x)$ is integrable in $(-\infty,\, \infty)$ for some N, then the generalized function $f(x)$ equivalent to the ordinary function is defined as any sequence of good functions $\{f_n(x)\}$ such that, for any good function $g(x)$,

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\, g(x)\, dx = \int_{-\infty}^{\infty} f(x)\, g(x)\, dx \tag{2.4.3}$$

For example, the generalized function equivalent to zero can be represented by either of the sequences $\left\{\frac{\phi(x)}{n}\right\}$ and $\left\{\frac{\phi(x)}{n^2}\right\}$.

The unit function, $I(x)$, is defined by

$$\int_{-\infty}^{\infty} I(x)\, g(x)\, dx = \int_{-\infty}^{\infty} g(x)\, dx \tag{2.4.4}$$

for any good function $g(x)$. A very important and useful good function that defines the unit function is $\left\{\exp\left(-\frac{x^2}{4n}\right)\right\}$. Thus, the unit function is the generalized function that is equivalent to the ordinary function $f(x) = 1$.

The *Heaviside function*, $H(x)$, is defined by

$$\int_{-\infty}^{\infty} H(x)\, g(x)\, dx = \int_{0}^{\infty} g(x)\, dx. \tag{2.4.5}$$

The generalized function $H(x)$ is equivalent to the ordinary unit function

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases} \tag{2.4.6}$$

since generalized functions are defined through the action on integrals of good functions, the value of $H(x)$ at $x = 0$ does not have significance here.

The *sign function*, sgn(x), is defined by

$$\int_{-\infty}^{\infty} \mathrm{sgn(x)\, g(x)\, dx} = \int_{0}^{\infty} g(x)\, dx - \int_{-\infty}^{0} g(x)\, dx \tag{2.4.7}$$

for any good function $g(x)$. Thus, sgn(x) can be identified with the ordinary function

$$\mathrm{sgn(x)} = \begin{cases} -1, & x < 0, \\ +1, & x > 0. \end{cases} \tag{2.4.8}$$

In fact, $\mathrm{sgn(x) = 2\, H(x) - I(x)}$ can be seen as follows:

$$\begin{aligned}
\int_{-\infty}^{\infty} \mathrm{sgn(x)\, g(x)\, dx} &= \int_{-\infty}^{\infty} \left[2\, H(x) - I(x)\right] g(x)\, dx \\
&= 2 \int_{-\infty}^{\infty} H(x)\, g(x)\, dx - \int_{-\infty}^{\infty} I(x)\, g(x)\, dx \\
&= 2 \int_{0}^{\infty} g(x)\, dx - \int_{-\infty}^{\infty} g(x)\, dx \\
&= \int_{0}^{\infty} g(x)\, dx - \int_{-\infty}^{0} g(x)\, dx
\end{aligned}$$

In 1926, Dirac introduced the delta function, $\delta(x)$, having the following properties

$$\begin{aligned} \delta(x) = 0, \qquad x \neq 0, \\ \int\limits_{-\infty}^{\infty} \delta(x) dx = 1. \end{aligned} \tag{2.4.9}$$

The Dirac delta function, $\delta(x)$ is defined so that for any good function $\phi(x)$,

$$\int_{-\infty}^{\infty} \delta(x)\,\phi(x)\,dx = \phi(0).$$

There is no ordinary function equivalent to the delta function.

The properties (2.4.9) cannot be satisfied by any ordinary functions in classical mathematics. Hence, the delta function is not a function in the classical sense like an ordinary function $f(x)$, $\delta(x)$ is not a value of δ at x. However, it can be treated as a function in the generalized sense, and in fact, $\delta(x)$ is called a *generalized function* or *distribution*. The concept of the delta function is clear and simple in modern mathematics. It is very useful in physics and engineering. Physically, the delta function represents a point mass, that is a particle of unit mass located at the origin. In this context, it may be called a *mass-density* function. This leads to the result for a point particle that can be considered as the limit of a sequence of continuous distributions which become more and more concentrated. Even though $\delta(x)$ is not a function in the classical sense, it can be approximated by a sequence of ordinary functions. As an example, we consider the sequence of functions

$$\delta_n(x) = \sqrt{\frac{n}{\pi}}\,\exp(-nx^2), \quad n = 1, 2, 3, \ldots. \tag{2.4.10}$$

Clearly, $\delta_n(x) \to 0$ as $n \to \infty$ for any $x \neq 0$ and $\delta_n(0) \to \infty$ as $n \to \infty$ as shown in Figure 2.4. Also, for all $n = 1, 2, 3, \ldots$,

$$\int_{-\infty}^{\infty} \delta_n(x) dx = 1$$

and

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n(x) dx = \int_{-\infty}^{\infty} \delta(x) dx = 1$$

as expected. So the delta function can be considered as the limit of a sequence of ordinary functions, and we write

$$\delta(x) = \lim_{n\to\infty} \sqrt{\frac{n}{\pi}}\exp(-nx^2). \tag{2.4.11}$$

Sometimes, the delta function $\delta(x)$ is defined by its fundamental property

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)\,dx = f(a), \tag{2.4.12}$$

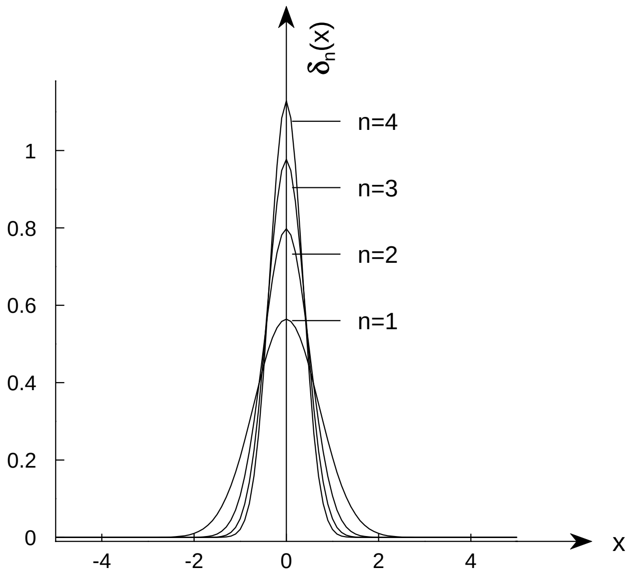

Figure 2.4 The sequence of delta functions, $\delta_n(x)$.

where $f(x)$ is continuous in any interval containing the point $x = a$. Clearly,

$$\int_{-\infty}^{\infty} f(a)\delta(x-a)\,dx = f(a)\int_{-\infty}^{\infty} \delta(x-a)\,dx = f(a). \tag{2.4.13}$$

Thus, (2.4.12) and (2.4.13) lead to the result

$$f(x)\delta(x-a) = f(a)\delta(x-a). \tag{2.4.14}$$

The following results are also true

$$x\,\delta(x) = 0 \tag{2.4.15}$$

$$\delta(x-a) = \delta(a-x). \tag{2.4.16}$$

Result (2.4.16) shows that $\delta(x)$ is an even function.

Clearly, the result

$$\int_{-\infty}^{x} \delta(y)\,dy = \begin{Bmatrix} 1, & x > 0 \\ 0, & x < 0 \end{Bmatrix} = H(x)$$

shows that

$$\frac{d}{dx}H(x) = \delta(x). \tag{2.4.17}$$

The Fourier transform of the Dirac delta function is

$$\mathscr{F}\{\delta(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \delta(x)\, dx = \frac{1}{\sqrt{2\pi}}. \tag{2.4.18}$$

Hence,

$$\delta(x) = \mathscr{F}^{-1}\left\{\frac{1}{\sqrt{2\pi}}\right\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx}\, dk. \tag{2.4.19}$$

This is an integral representation of the *delta function* extensively used in quantum mechanics. Also, (2.4.19) can be rewritten as

$$\delta(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx}\, dx. \tag{2.4.20}$$

The Dirac delta function, $\delta(x)$ is defined so that for any good function $g(x)$,

$$\langle \delta,\, g\rangle = \int_{-\infty}^{\infty} \delta(x)\, g(x)\, dx = g(0). \tag{2.4.21}$$

Derivatives of generalized functions are defined by the derivatives of any equivalent sequences of good functions. We can integrate by parts using any member of the sequences and assuming $g(x)$ vanishes at infinity. We can obtain this definition as follows:

$$\begin{aligned} \langle f',\, g\rangle &= \int_{-\infty}^{\infty} f'(x)\, g(x)\, dx \\ &= [f(x)\, g(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)\, g'(x)\, dx = -\langle f,\, g'\rangle . \end{aligned}$$

The derivative of a generalized function f is the generalized function f' defined by

$$\langle f',\, g\rangle = -\langle f,\, g'\rangle \tag{2.4.22}$$

for any good function g.

The differential calculus of generalized functions can easily be developed with locally integrable functions. To every locally integrable function f, there corresponds a *generalized function* (or *distribution*) defined by

$$\langle f,\, \phi\rangle = \int_{-\infty}^{\infty} f(x)\, \phi(x)\, dx \tag{2.4.23}$$

where ϕ is a test function in $\mathbb{R} \to \mathbb{C}$ with bounded support (ϕ is infinitely differentiable with its derivatives of all orders exist and are continuous).

The derivative of a generalized function f is the generalized function f' defined by

$$\langle f',\, \phi\rangle = -\,\langle f,\, \phi'\rangle \tag{2.4.24}$$

for all test functions ϕ. This definition follows from the fact that

$$\begin{aligned}\langle f',\, \phi\rangle &= \int_{-\infty}^{\infty} f'(x)\,\phi(x)\,dx \\ &= [f(x)\,\phi(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)\,\phi'(x)\,dx = -\,\langle f,\, \phi'\rangle\end{aligned}$$

which was obtained from integration by parts and using the fact that ϕ vanishes at infinity.

It is easy to check that $H'(x) = \delta(x)$, for

$$\begin{aligned}\langle H',\, \phi\rangle &= \int_{-\infty}^{\infty} H'(x)\,\phi(x)\,dx = -\int_{-\infty}^{\infty} H(x)\,\phi'(x)\,dx \\ &= -\int_{0}^{\infty} \phi'(x)\,dx = -\,[\phi(x)]_0^{\infty} = \phi(0) = \langle \delta,\, \phi\rangle\,.\end{aligned}$$

Another result is

$$\langle \delta',\, \phi\rangle = -\int_{-\infty}^{\infty} \delta(x)\,\phi'(x)\,dx = -\phi'(0)\,.$$

It is easy to verify

$$f(x)\,\delta(x) = f(0)\,\delta(x)\,.$$

We next define $|x| = x\,\mathrm{sgn(x)}$ and calculate its derivative as follows. We have

$$\begin{aligned}\frac{d}{dx}|x| = \frac{d}{dx}\,\{x\,\mathrm{sgn(x)}\} &= x\,\frac{d}{dx}\,\{\mathrm{sgn(x)}\} + \mathrm{sgn(x)}\,\frac{\mathrm{dx}}{\mathrm{dx}} \\ &= x\,\frac{d}{dx}\,\{2\,H(x) - I(x)\} + \mathrm{sgn(x)} \\ &= 2x\,\delta(x) + \mathrm{sgn(x)} = \mathrm{sgn(x)}\end{aligned} \tag{2.4.25}$$

which is, by $\mathrm{sgn(x)} = 2\,\mathrm{H(x)} - \mathrm{I(x)}$ and $x\,\delta(x) = 0$.

Similarly, we can show that

$$\frac{d}{dx}\,\{\mathrm{sgn(x)}\} = 2H'(x) = 2\delta(x). \tag{2.4.26}$$

If we can show that (2.3.1) holds for good functions, it follows that it holds for generalized functions.

THEOREM 2.4.1 The Fourier transform of a good function is a good function.

PROOF The Fourier transform of a good function $f(x)$ exists and is given by

$$\mathscr{F}\{f(x)\} = F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx. \tag{2.4.27}$$

Differentiating $F(k)$ n times and integrating N times by parts, we get

$$\begin{aligned} \left|F^{(n)}(k)\right| &\le \left| \frac{(-1)^N}{(-ik)^N} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \frac{d^N}{dx^N} \{(-ix)^n f(x)\}\, dx \right| \\ &\le \frac{1}{|k|^N} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left| \frac{d^N}{dx^N} \{x^n f(x)\} \right| dx. \end{aligned}$$

Evidently, all derivatives tend to zero as fast as $|k|^{-N}$ as $|k| \to \infty$ for any $N > 0$ and hence, $F(k)$ is a good function. ∎

THEOREM 2.4.2 If $f(x)$ is a good function with the Fourier transform (2.4.27), then the inverse Fourier transform is given by

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} F(k)\, dk. \tag{2.4.28}$$

PROOF For any $\epsilon > 0$, we have

$$\mathscr{F}\left\{e^{-\epsilon x^2} F(-x)\right\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx - \epsilon x^2} \left\{ \int_{-\infty}^{\infty} e^{ixt} f(t)\, dt \right\} dx.$$

Since f is a good function, the order of integration can be interchanged to obtain

$$\mathscr{F}\left\{e^{-\epsilon x^2} F(-x)\right\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t)\, dt \int_{-\infty}^{\infty} e^{-i(k-t)x - \epsilon x^2}\, dx$$

which is, by similar calculation used in Example 2.3.1,

$$= \frac{1}{\sqrt{4\pi\epsilon}} \int_{-\infty}^{\infty} \exp\left[-\frac{(k-t)^2}{4\epsilon}\right] f(t)\, dt\,.$$

Using the fact that

$$\frac{1}{\sqrt{4\pi\epsilon}} \int_{-\infty}^{\infty} \exp\left[-\frac{(k-t)^2}{4\epsilon}\right] dt = 1,$$

we can write

$$\mathscr{F}\left\{e^{-\epsilon x^2} F(-x)\right\} - f(k)\,.\,1$$
$$= \frac{1}{\sqrt{4\pi\epsilon}} \int_{-\infty}^{\infty} [f(t) - f(k)] \exp\left[-\frac{(k-t)^2}{4\epsilon}\right] dt. \quad (2.4.29)$$

Since f is a good function, we have

$$\left|\frac{f(t)-f(k)}{t-k}\right| \le \max_{x\in\mathbb{R}} |f'(x)|\,.$$

It follows from (2.4.29) that

$$\left|\mathscr{F}\left\{e^{-\epsilon x^2} F(-x)\right\} - f(k)\right|$$
$$\le \frac{1}{\sqrt{4\pi\epsilon}} \max_{x\in\mathbb{R}} |f'(x)| \int_{-\infty}^{\infty} |t-k| \exp\left[-\frac{(t-k)^2}{4\epsilon}\right] dt$$
$$= \frac{1}{\sqrt{4\pi\epsilon}} \max_{x\in\mathbb{R}} |f'(x)|\; 4\epsilon \int_{-\infty}^{\infty} |\alpha|\; e^{-\alpha^2} d\alpha \to 0$$

as $\epsilon \to 0$, where $\alpha = \frac{t-k}{2\sqrt{\epsilon}}$.
Consequently,

$$f(k) = \mathscr{F}\{F(-x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} F(-x)\, dx$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} F(x)\, dx$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx}\, dx \int_{-\infty}^{\infty} e^{-i\xi x} f(\xi)\, d\xi.$$

Interchanging k with x, this reduces to the Fourier integral formula (2.2.4) and hence, the theorem is proved. ∎

Example 2.4.1 The Fourier transform of a constant function c is

$$\mathscr{F}\{c\} = \sqrt{2\pi}.c.\delta(k). \quad (2.4.30)$$

In the ordinary sense

$$\mathscr{F}\{c\} = \frac{c}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx}\, dx$$

is not a well defined (divergent) integral. However, treated as a generalized function, $c = c\,I(x)$ and we consider $\left\{\exp\left(-\frac{x^2}{4n}\right)\right\}$ as an equivalent sequence to the unit function, $I(x)$. Thus,

$$\mathscr{F}\left\{c\;\exp\left(-\frac{x^2}{4n}\right)\right\} = \frac{c}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-ikx - \frac{x^2}{4n}\right) dx$$

which is, by Example 2.3.1,

$$= c\sqrt{2n}\,\exp(-nk^2) = \sqrt{2\pi}.c.\sqrt{\frac{n}{\pi}}\,\exp(-nk^2)$$
$$= \sqrt{2\pi}.c.\delta_n(k) = \sqrt{2\pi}.c.\delta(k) \quad \text{as} \quad n \to \infty,$$

since $\{\delta_n(k)\} = \left\{\sqrt{\frac{n}{\pi}}\exp\left(-nk^2\right)\right\}$ is a sequence equivalent to the delta function defined by (2.4.10).

□

Example 2.4.2 Show that

$$\mathscr{F}\{e^{-ax}H(x)\} = \frac{1}{\sqrt{2\pi}(ik+a)}, \quad a > 0. \tag{2.4.31}$$

We have, by definition,

$$\mathscr{F}\{e^{-ax}H(x)\} = \frac{1}{\sqrt{2\pi}}\int_0^\infty \exp\{-x(ik+a)\}dx = \frac{1}{\sqrt{2\pi}(ik+a)}.$$

□

Example 2.4.3 By considering the function (see Figure 2.5)

$$f_a(x) = e^{-ax}H(x) - e^{ax}H(-x), \quad a > 0, \tag{2.4.32}$$

find the Fourier transform of sgn(x). In Figure 2.5, the vertical axis (y-axis) represents $f_a(x)$ and the horizontal axis represents the x-axis.

We have, by definition,

$$\begin{aligned}\mathscr{F}\{f_a(x)\} &= -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^0 \exp\{(a-ik)x\}dx \\ &\quad + \frac{1}{\sqrt{2\pi}}\int_0^\infty \exp\{-(a+ik)x\}dx \\ &= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{a+ik} - \frac{1}{a-ik}\right] = \sqrt{\frac{2}{\pi}}\cdot\frac{(-ik)}{a^2+k^2}.\end{aligned}$$

In the limit as $a \to 0$, $f_a(x) \to \text{sgn(x)}$ and then

$$\mathscr{F}\{\text{sgn(x)}\} = \sqrt{\frac{2}{\pi}}\cdot\frac{1}{\text{ik}}.$$

Or,

$$\mathscr{F}\left\{\sqrt{\frac{\pi}{2}}\,i\,\text{sgn(x)}\right\} = \frac{1}{k}.$$

□

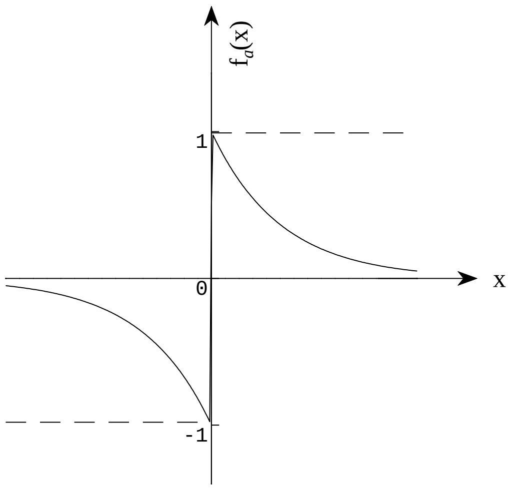

Figure 2.5 Graph of the function $f_a(x)$.

Example 2.4.4 (Fourier Integral Theorem)

Using the delta function representation (2.4.12) of a continuous function $f(x)$, we give a short proof of the Fourier integral theorem (2.2.4). We have, by (2.4.12) and (2.4.19),

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} f(\xi)\delta(x-\xi)d\xi \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)d\xi \int_{-\infty}^{\infty} e^{ik(x-\xi)}dk \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\left\{\int_{-\infty}^{\infty} e^{-ik\xi}f(\xi)d\xi\right\}dk. \end{aligned} \tag{2.4.33}$$

This is the desired Fourier integral theorem (2.2.4). □

2.5 Basic Properties of Fourier Transforms

THEOREM 2.5.1 If $\mathscr{F}\{f(x)\} = F(k)$, then

(a) (Shifting)	$\mathscr{F}\{f(x-a)\} = e^{-ika}F(k),$	(2.5.1)
(b) (Scaling)	$\mathscr{F}\{f(ax \pm b)\} = \frac{1}{\lvert a \rvert}\, e^{\pm \frac{ibk}{a}} F(\frac{k}{a}), \quad a \neq 0$	(2.5.2)
(c) (Conjugate)	$\mathscr{F}\{\overline{f(-x)}\} = \overline{\mathscr{F}\{f(x)\}},$	(2.5.3)
(d) (Translation)	$\mathscr{F}\{e^{iax}f(x)\} = F(k-a),$	(2.5.4)
(e) (Duality)	$\mathscr{F}\{F(x)\} = f(-k),$	(2.5.5)
(f) (Composition)	$\int_{-\infty}^{\infty} F(k)g(k)e^{ikx}dk = \int_{-\infty}^{\infty} f(\xi)G(\xi - x)d\xi,$ where $G(k) = \mathscr{F}\{g(x)\},$	(2.5.6)
(g) (Modulation)	$\mathscr{F}\{f(x)\cos ax\} = \frac{1}{2}\,[F(k-a) + F(k+a)]$ $\mathscr{F}\{f(x)\sin ax\} = \frac{1}{2i}\,[F(k-a) - F(k+a)].$	

PROOF (a) We obtain, from the definition,

$$\begin{aligned}\mathscr{F}\{f(x-a)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x-a)dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ik(\xi+a)} f(\xi)d\xi, \qquad (x - a = \xi) \\ &= e^{-ika}\mathscr{F}\{f(x)\}.\end{aligned}$$

The proofs of results (b)–(d) follow easily from the definition of the Fourier transform. We give a proof of the duality (e) and composition (f).

We have, by definition,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} F(k)dk = \mathscr{F}^{-1}\{F(k)\}.$$

Interchanging x and k, and then replacing k by $-k$, we obtain

$$f(-k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} F(x)dx = \mathscr{F}\{F(x)\}.$$

To prove (f), we have

$$\begin{aligned}\int_{-\infty}^{\infty} F(k)g(k)\, e^{ikx} dk &= \int_{-\infty}^{\infty} g(k)\, e^{ikx} dk \;\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ik\xi} f(\xi)d\xi \\ &= \int_{-\infty}^{\infty} f(\xi)d\xi \;\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ik(\xi-x)} g(k)dk \\ &= \int_{-\infty}^{\infty} f(\xi)G(\xi - x)d\xi.\end{aligned}$$

In particular, when $x = 0$, (2.5.6) reduces to

$$\int_{-\infty}^{\infty} F(k)g(k)dk = \int_{-\infty}^{\infty} f(\xi)G(\xi)d\xi.$$

This is known as the composition rule which can readily be proved. ∎

THEOREM 2.5.2 If $f(x)$ is piecewise continuously differentiable and absolutely integrable, then

(i) $F(k)$ is bounded for $-\infty < k < \infty$,

(ii) $F(k)$ is continuous for $-\infty < k < \infty$.

PROOF It follows from the definition that

$$\begin{aligned}|F(k)| &\le \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |e^{-ikx}||f(x)|dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)|dx = \frac{c}{\sqrt{2\pi}},\end{aligned}$$

where $c = \int_{-\infty}^{\infty} |f(x)|dx =$ constant. This proves result (i).

To prove (ii), we have

$$\begin{aligned}|F(k+h)-F(k)| &\le \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|e^{-ihx}-1||f(x)|dx \\ &\le \sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}|f(x)|dx.\end{aligned}$$

Since $\lim_{h\to 0}|e^{-ihx}-1|=0$ for all $x\in\mathbb{R}$, we obtain

$$\lim_{h\to 0}|F(k+h)-F(k)|\le \lim_{h\to 0}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|e^{-ihx}-1||f(x)|dx=0.$$

This shows that $F(k)$ is continuous. ∎

THEOREM 2.5.3 ***(****Riemann-Lebesgue Lemma****).***
If $F(k)=\mathscr{F}\{f(x)\}$, then

$$\lim_{|k|\to\infty}|F(k)|=0. \tag{2.5.7}$$

PROOF Since $e^{-ikx}=-e^{-ikx-i\pi}$, we have

$$\begin{aligned}F(k) &= -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ik(x+\frac{\pi}{k})}f(x)dx \\ &= -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ikx}f\left(x-\frac{\pi}{k}\right)dx.\end{aligned}$$

Hence,

$$\begin{aligned}F(k) &= \frac{1}{2}\left\{\frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty}e^{-ikx}f(x)dx-\int_{-\infty}^{\infty}e^{-ikx}f\left(x-\frac{\pi}{k}\right)dx\right]\right\} \\ &= \frac{1}{2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ikx}\left[f(x)-f\left(x-\frac{\pi}{k}\right)\right]dx.\end{aligned}$$

Therefore,

$$|F(k)|\le \frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty}\left|f(x)-f\left(x-\frac{\pi}{k}\right)\right|dx.$$

Thus, we obtain

$$\lim_{|k|\to\infty} |F(k)| \le \frac{1}{2\sqrt{2\pi}} \lim_{|k|\to\infty} \int_{-\infty}^{\infty} \left| f(x) - f\left(x - \frac{\pi}{k}\right) \right| dx = 0.$$

∎

THEOREM 2.5.4 If $f(x)$ is continuously differentiable and $f(x) \to 0$ as $|x| \to \infty$, then

$$\mathscr{F}\{f'(x)\} = (ik)\mathscr{F}\{f(x)\} = ik\, F(k). \tag{2.5.8}$$

PROOF We have, by definition,

$$\mathscr{F}\{f'(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f'(x) dx$$

which is, integrating by parts,

$$= \frac{1}{\sqrt{2\pi}} \left[f(x) e^{-ikx} \right]_{-\infty}^{\infty} + \frac{ik}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x) dx$$

$$= (ik) F(k).$$

If $f(x)$ is continuously n-times differentiable and $f^{(k)}(x) \to 0$ *as* $|x| \to \infty$ for $k = 1, 2, \ldots, (n-1)$, then the Fourier transform of the nth derivative is

$$\mathscr{F}\{f^{(n)}(x)\} = (ik)^n \mathscr{F}\{f(x)\} = (ik)^n F(k). \tag{2.5.9}$$

A repeated application of Theorem 2.5.4 to higher derivatives gives the result.

The operational results similar to those of (2.5.8) and (2.5.9) hold for partial derivatives of a function of two or more independent variables. For example, if $u(x,t)$ is a function of space variable x and time variable t, then

$$\mathscr{F}\left\{\frac{\partial u}{\partial x}\right\} = ik\, U(k,t), \quad \mathscr{F}\left\{\frac{\partial^2 u}{\partial x^2}\right\} = -k^2\, U(k,t),$$

$$\mathscr{F}\left\{\frac{\partial u}{\partial t}\right\} = \frac{dU}{dt}, \quad \mathscr{F}\left\{\frac{\partial^2 u}{\partial t^2}\right\} = \frac{d^2 U}{dt^2},$$

where $U(k,t) = \mathscr{F}\{u(x,t)\}$. ∎

DEFINITION 2.5.1 *The convolution of two integrable functions $f(x)$ and $g(x)$, denoted by $(f * g)(x)$, is defined by*

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x - \xi) g(\xi) d\xi, \tag{2.5.10}$$

provided the integral in (2.5.10) exists, where the factor $\frac{1}{\sqrt{2\pi}}$ is a matter of choice. In the study of convolution, this factor is often omitted as this factor does not affect the properties of convolution. We will include or exclude the factor $\frac{1}{\sqrt{2\pi}}$ freely in this book.

We give some examples of convolution.

Example 2.5.1 Find the convolution of
(a) $f(x)=\cos x$ and $g(x)=\exp(-a|x|)$, $a>0$,
(b) $f(x)=\chi_{[a,b]}(x)$ and $g(x)=x^2$,
where $\chi_{[a,b]}(x)$ is the characteristic function of the interval $[a,b]\subseteq\mathbb{R}$ defined by

$$\chi_{[a,b]}(x)=\left\{\begin{array}{ll}1, & a\le x\le b\\ 0, & \text{otherwise}\end{array}\right\}.$$

(a) We have, by definition,

$$\begin{aligned}(f*g)(x) &= \int_{-\infty}^{\infty} f(x-\xi)\ g(\xi)d\xi = \int_{-\infty}^{\infty}\cos(x-\xi)\ e^{-a|\xi|}d\xi\\ &= \int_{-\infty}^{0}\cos(x-\xi)\ e^{a\xi}d\xi + \int_{0}^{\infty}\cos(x-\xi)\ e^{-a\xi}d\xi\\ &= \int_{0}^{\infty}\cos(x+\xi)\ e^{-a\xi}d\xi + \int_{0}^{\infty}\cos(x-\xi)\ e^{-a\xi}d\xi\\ &= 2\cos x\int_{0}^{\infty}\cos\xi\ e^{-a\xi}d\xi \quad = \quad \frac{2a\cos x}{(1+a^2)}.\end{aligned}$$

If $a=1$, then $f*g)(x)=f(x)$ so that g becomes an identity element of convolution. The question is whether it is true for all $g(x)$.

(b) We have

$$\begin{aligned}(f*g)(x) &= \int_{-\infty}^{\infty} f(x-\xi)\ g(\xi)d\xi = \int_{-\infty}^{\infty}\chi_{[a,b]}(x-\xi)\ g(\xi)d\xi\\ &= \int_{-\infty}^{\infty}\chi_{[a,b]}(\xi)\ g(x-\xi)d\xi = \int_a^b g(x-\xi)\ d\xi = \int_a^b (x-\xi)^2\ d\xi\\ &= \frac{1}{3}\left\{(x-a)^3-(x-b)^3\right\}.\end{aligned}$$

∎

THEOREM 2.5.5 ***(****Convolution Theorem****).***
If $\mathscr{F}\{f(x)\} = F(k)$ and $\mathscr{F}\{g(x)\} = G(k)$, then

$$\mathscr{F}\{f(x) * g(x)\} = F(k)G(k), \tag{2.5.11}$$

or,

$$f(x) * g(x) = \mathscr{F}^{-1}\{F(k)G(k)\}, \tag{2.5.12}$$

or, equivalently,

$$\int_{-\infty}^{\infty} f(x-\xi)g(\xi)d\xi = \int_{-\infty}^{\infty} e^{ikx}F(k)G(k)dk. \tag{2.5.13}$$

PROOF We have, by the definition of the Fourier transform,

$$\begin{aligned}
\mathscr{F}\{f(x) * g(x)\} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}dx \int_{-\infty}^{\infty} f(x-\xi)g(\xi)d\xi \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ik\xi}g(\xi)d\xi \int_{-\infty}^{\infty} e^{-ik(x-\xi)}f(x-\xi)dx \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ik\xi}g(\xi)d\xi \int_{-\infty}^{\infty} e^{-ik\eta}f(\eta)d\eta = G(k)F(k),
\end{aligned}$$

where, in this proof, the factor $\frac{1}{\sqrt{2\pi}}$ is included in the definition of the convolution and all necessary interchanges of the order of integration are valid. This completes the proof. ∎

The convolution has the following algebraic properties:

$$f * g = g * f \quad \text{(Commutative)}, \tag{2.5.14}$$

$$f * (g * h) = (f * g) * h \quad \text{(Associative)}, \tag{2.5.15}$$

$$(\alpha f + \beta g) * h = \alpha\,(f * h) + \beta\,(g * h) \quad \text{(Distributive)}, \tag{2.5.16}$$

$$f * \sqrt{2\pi}\delta = f = \sqrt{2\pi}\delta * f \quad \text{(Identity)}, \tag{2.5.17}$$

where α and β are constants.

We give proofs of (2.5.15) and (2.5.16). If $f * (g * h)$ exists, then

$$\begin{aligned}
[f * (g * h)]\,(x) &= \int_{-\infty}^{\infty} f(x-\xi)(g * h)(\xi)d\xi \\
&= \int_{-\infty}^{\infty} f(x-\xi) \int_{-\infty}^{\infty} g(\xi - t)h(t)\,dt \;\; d\xi
\end{aligned}$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x-\xi)\, g(\xi - t) d\xi \right] h(t) dt$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x-t-\eta) \;\; g(\eta) d\eta \right] h(t) dt \quad (\text{put } \xi - t = \eta)$$

$$= \int_{-\infty}^{\infty} \left[(f * g)\,(x-t) \right] h(t) dt$$

$$= \left[(f * g) * h \right](x),$$

where, in the above proof, under suitable assumptions, the interchange of the order of integration can be justified.

Similarly, we prove (2.5.16) using the right-hand side of (2.5.16), that is,

$$\alpha\,(f * h) + \beta\,(g * h) = \alpha \int_{-\infty}^{\infty} f(x-\xi) h(\xi) d\xi + \beta \int_{-\infty}^{\infty} g(x-\xi) h(\xi) d\xi$$

$$= \int_{-\infty}^{\infty} \left[\alpha f(x-\xi) + \beta g(x-\xi) \right] h(\xi) d\xi$$

$$= \left[(\alpha f + \beta g) * h \right](x).$$

Another proof of the associative property of the convolution is given below.

We apply the Fourier transform to the left-hand side of (2.5.15) and then use the convolution theorem (2.5.5) so that

$$\begin{aligned} \mathscr{F}\{f * (g * h)\} &= \mathscr{F}\{f\}\,\mathscr{F}\{(g * h)\} \\ &= F(k)\left[\mathscr{F}\{g\}\,\mathscr{F}\{h\}\right] \\ &= F(k)\left[G(k)H(k)\right] \\ &= \left[F(k)G(k)\right]H(k) \\ &= \mathscr{F}\{(f * g)\}\,\mathscr{F}\{h\} \\ &= \mathscr{F}\{(f * g) * h\}. \end{aligned}$$

Applying the $\mathscr{F}^{-1}$ on both sides, we obtain

$$f * (g * h) = (f * g) * h.$$

Similarly, all properties (2.5.14)–(2.5.17) of convolution can easily be proved using the convolution theorem 2.5.5.

In view of the commutative property (2.5.14) of the convolution, (2.5.13) can be written as

$$\int_{-\infty}^{\infty} f(\xi) g(x-\xi) d\xi = \int_{-\infty}^{\infty} e^{ikx} F(k) G(k) dk. \qquad (2.5.18)$$

This is valid for all real x, and hence, putting $x = 0$ gives

$$\int_{-\infty}^{\infty} f(\xi)g(-\xi)d\xi = \int_{-\infty}^{\infty} f(x)g(-x)dx = \int_{-\infty}^{\infty} F(k)G(k)dk. \tag{2.5.19}$$

We substitute $g(x) = \overline{f(-x)}$ to obtain

$$G(k) = \mathscr{F}\{g(x)\} = \mathscr{F}\left\{\overline{f(-x)}\right\} = \overline{\mathscr{F}\{f(x)\}} = \overline{F(k)}.$$

Evidently, (2.5.19) becomes

$$\int_{-\infty}^{\infty} f(x)\,\overline{f(x)}dx = \int_{-\infty}^{\infty} F(k)\,\overline{F(k)}dk \tag{2.5.20}$$

or,

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(k)|^2 dk. \tag{2.5.21}$$

This is well known as *Parseval's relation.*

For square integrable functions $f(x)$ and $g(x)$, the *inner product* $\langle f, g\rangle$ is defined by

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\,\overline{g(x)}dx \tag{2.5.22}$$

so the *norm* $\|f\|_2$ is defined by

$$\|f\|_2^2 = \langle f, f\rangle = \int_{-\infty}^{\infty} f(x)\,\overline{f(x)}dx = \int_{-\infty}^{\infty} |f(x)|^2 dx. \tag{2.5.23}$$

The function space $L^2(\mathbb{R})$ of all complex-valued Lebesgue square integrable functions with the inner product defined by (2.5.22) is a complete normed space with the norm (2.5.23). In terms of the norm, the Parseval relation takes the form

$$\|f\|_2 = \|F\|_2 = \|\mathscr{F} f\|_2. \tag{2.5.24}$$

This means that the Fourier transform action is *unitary*. Physically, the quantity $\|f\|_2$ is a measure of energy and $\|F\|_2$ represents the *power spectrum* of f.

THEOREM 2.5.6 ***(****General Parseval's Relation****).***
If $\mathscr{F}\{f(x)\} = F(k)$ and $\mathscr{F}\{g(x)\} = G(k)$ then

$$\int_{-\infty}^{\infty} f(x)\,\overline{g(x)}dx = \int_{-\infty}^{\infty} F(k)\,\overline{G(k)}dk. \tag{2.5.25}$$

PROOF We proceed formally to obtain

$$\int_{-\infty}^{\infty} F(k)\,\overline{G(k)}dk = \int_{-\infty}^{\infty} dk \cdot \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iky}f(y)\ dy\,\overline{\int_{-\infty}^{\infty} e^{-ikx}g(x)\ dx}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(y)\ dy\int_{-\infty}^{\infty}\overline{g(x)}dx\int_{-\infty}^{\infty} e^{ik(x-y)}dk$$

$$= \int_{-\infty}^{\infty}\overline{g(x)}\ dx\int_{-\infty}^{\infty}\delta(x-y)f(y)\ dy = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\ dx.$$

In particular, when $g(x)=f(x)$, the above result agrees with (2.5.20).

Second Proof of (2.5.25).

Using the inverse Fourier transform, we have

$$f(x)\overline{g(x)} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}F(k)dk\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\xi x}\,\overline{G(\xi)}\,d\xi$$

$$= \int_{-\infty}^{\infty} F(k)dk\,\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(k-\xi)x}\,\overline{G(\xi)}\,d\xi.$$

Thus,

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}dx = \int_{-\infty}^{\infty} F(k)dk\int_{-\infty}^{\infty}\delta(k-\xi)\,\overline{G(\xi)}\,d\xi$$

$$= \int_{-\infty}^{\infty} F(k)\,\overline{G(k)}\,dk.$$

We now use an indirect method to obtain the Fourier transform of sgn(x), that is,

$$\mathscr{F}\{\mathrm{sgn(x)}\} = \sqrt{\frac{2}{\pi}}\,\frac{1}{\mathrm{ik}}. \tag{2.5.26}$$

From (2.4.26), we find

$$\mathscr{F}\left\{\frac{d}{dx}\mathrm{sgn(x)}\right\} = \mathscr{F}\{2H'(x)\} = 2\mathscr{F}\{\delta(x)\} = \sqrt{\frac{2}{\pi}},$$

which is, by (2.5.8),

$$ik\ \mathscr{F}\{\mathrm{sgn(x)}\} = \sqrt{\frac{2}{\pi}},$$

or

$$\mathscr{F}\{\text{sgn(x)}\} = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\text{ik}}.$$

The Fourier transform of $H(x)$ follows from (2.4.30) and (2.5.26):

$$\begin{aligned} \mathscr{F}\{H(x)\} &= \frac{1}{2}\mathscr{F}\{1 + \text{sgn(x)}\} = \frac{1}{2}[\mathscr{F}\{1\} + \mathscr{F}\{\text{sgn(x)}\}] \\ &= \sqrt{\frac{\pi}{2}}\left[\delta(k) + \frac{1}{i\pi k}\right]. \end{aligned} \tag{2.5.27}$$

$\blacksquare$

2.6 Poisson's Summation Formula

A class of functions designated as $L^p(\mathbb{R})$ is of great importance in the theory of Fourier transformations, where $p(\geq 1)$ is any real number. We denote the vector space of all complex-valued functions $f(x)$ of the real variable x. If f is a locally integrable function such that $|f|^p \in L(\mathbb{R})$, then we say f is p-th power Lebesgue integrable. The set of all such functions is written $L^p(\mathbb{R})$. The number $||f||_p$ is called the L^p-norm of f and is defined by

$$||f||_p = \left[\int_{-\infty}^{\infty} |f(x)|^p \, dx\right]^{\frac{1}{p}} < \infty. \tag{2.6.1}$$

Suppose f is a Lebesgue integrable function on $\mathbb{R}$. Since $\exp(-ikx)$ is continuous and bounded, the product $\exp(-ikx)\, f(x)$ is locally integrable for any $k \in \mathbb{R}$. Also, $|\exp(-ikx)| \leq 1$ for all k and x on $\mathbb{R}$. Consider the inner product

$$\langle f, e^{ikx} \rangle = \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx, \qquad k \in \mathbb{R}. \tag{2.6.2}$$

Clearly,

$$\left|\int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx\right| \leq \int_{-\infty}^{\infty} |f(x)|\, dx = ||f||_1 < \infty. \tag{2.6.3}$$

This means that integral (2.6.2) exists for all $k \in \mathbb{R}$, and was used to define the Fourier transform, $F(k) = \mathscr{F}\{f(x)\}$ without the factor $\frac{1}{\sqrt{2\pi}}$.

Although the theory of Fourier series is a very important subject, a detailed study is beyond the scope of this book. Without rigorous analysis, we can establish a simple relation between the Fourier transform of functions in $L^1(\mathbb{R})$

and the Fourier series of related periodic functions in $L^1(-a,\, a)$ of period $2a$. If $f(x) \in L^1(-a,\, a)$ and is defined by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \qquad (-a \le x \le a), \tag{2.6.4}$$

where the Fourier coefficients c_n is given by

$$c_n = \frac{1}{2a} \int_{-a}^{a} f(x)\, e^{-ikx}\, dx. \tag{2.6.5}$$

THEOREM 2.6.1 If $f(x) \in L^1(\mathbb{R})$**,** then the series

$$\sum_{n=-\infty}^{\infty} f(x + 2na) \tag{2.6.6}$$

converges absolutely for almost all x in $(-a,\, a)$ and its sum $g(x) \in L^1(-a,\, a)$ with $g(x + 2a) = g(x)$ for $x \in \mathbb{R}$.

If a_n denotes the Fourier coefficient of a function g, then

$$a_n = \frac{1}{2a} \int_{-a}^{a} g(x)\, e^{-inx}\, dx = \frac{1}{2a} \int_{-\infty}^{\infty} f(x)\, e^{-inx}\, dx = \frac{1}{2a} F(n).$$

PROOF We have

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} \int_{-a}^{a} |f(x + 2na)|\, dx &= \lim_{N\to\infty} \sum_{n=-N}^{N} \int_{-a}^{a} |f(x + 2na)|\, dx \\
&= \lim_{N\to\infty} \sum_{n=-N}^{N} \int_{(2n-1)a}^{(2n+1)a} |f(t)|\, dt \\
&= \lim_{N\to\infty} \int_{-(2N+1)a}^{(2N+1)a} |f(t)|\, dt \\
&= \int_{-\infty}^{\infty} |f(t)|\, dt < \infty.
\end{aligned}$$

It follows from Lebesgue's theorem on monotone convergence that

$$\int_{-a}^{a} \left[\sum_{n=-\infty}^{\infty} |f(x + 2na)| \right] dx = \sum_{n=-\infty}^{\infty} \int_{-a}^{a} |f(x + 2na)|\, dx < \infty.$$

Hence, the series $\sum_{n=-\infty}^{\infty} f(x + 2na)$ converges absolutely for almost all x in $(-a,\, a)$. If $g_N(x) = \sum_{n=-N}^{N} f(x + 2na)$, $\lim_{N\to\infty} g_N(x) = g(x)$, where $g \in$

$\mathbb{L}^1(-a,\, a)$, and $g(x+2a)=g(x)$.

Moreover,

$$\begin{aligned}||g||_1 &= \int_{-a}^{a} |g(x)|\ dx = \int_{-a}^{a} \left| \sum_{n=-\infty}^{\infty} f(x+2na) \right| dx \\ &\leq \int_{-a}^{a} \sum_{n=-\infty}^{\infty} |f(x+2na)|\ dx \\ &= \sum_{n=-\infty}^{\infty} \int_{-a}^{a} |f(x+2na)|\ dx \\ &= \int_{-\infty}^{\infty} |f(x)|\ dx = ||f||_1.\end{aligned}$$

▮

We consider the Fourier series of $g(x)$ given by

$$g(x) = \sum_{m=-\infty}^{\infty} c_m \exp(im\pi x/a), \tag{2.6.7}$$

where the coefficients c_m for $m=0, \pm 1, \pm 2, ...$ are given by

$$c_m = \frac{1}{2a} \int_{-a}^{a} g(x) \exp(-im\pi x/a) dx. \tag{2.6.8}$$

We replace $g(x)$ by the limit of the sum

$$g(x) = \lim_{N\to\infty} \sum_{n=-N}^{N} f(x+2na), \tag{2.6.9}$$

so that (2.6.8) reduces to

$$\begin{aligned}c_m &= \frac{1}{2a} \lim_{N\to\infty} \sum_{n=-N}^{N} \int_{-a}^{a} f(x+2na) \exp(-im\pi x/a)\, dx \\ &= \frac{1}{2a} \lim_{N\to\infty} \sum_{n=-N}^{N} \int_{(2n-1)a}^{(2n+1)a} f(y) \exp(-im\pi y/a)\ \ dy \\ &= \frac{1}{2a} \lim_{N\to\infty} \int_{-(2N+1)a}^{(2N+1)a} f(x) \exp(-im\pi x/a)\ \ dx \\ &= \frac{\sqrt{2\pi}}{2a} F\left(\frac{m\pi}{a}\right),\end{aligned} \tag{2.6.10}$$

where $F\left(\frac{m\pi}{a}\right)$ is the discrete Fourier transform of $f(x)$.

Evidently,

$$\sum_{n=-\infty}^{\infty} f(x+2na) = g(x) = \sum_{n=-\infty}^{\infty} \frac{\sqrt{2\pi}}{2a} F\left(\frac{n\pi}{a}\right) \exp(in\pi x/a). \tag{2.6.11}$$

We let $x=0$ in (2.6.11) to obtain the *Poisson summation formula*

$$\sum_{n=-\infty}^{\infty} f(2na) = \sum_{n=-\infty}^{\infty} \frac{\sqrt{2\pi}}{2a} F\left(\frac{n\pi}{a}\right). \tag{2.6.12}$$

When $a=\pi$, this formula becomes an elegant form

$$\sum_{n=-\infty}^{\infty} f(2\pi n) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} F(n). \tag{2.6.13}$$

When $2a=1$, formula (2.6.12) becomes

$$\sum_{n=-\infty}^{\infty} f(n) = \sqrt{2\pi} \sum_{n=-\infty}^{\infty} F(2n\pi). \tag{2.6.14}$$

To obtain a more general formula, we assume that a is a given positive constant, and write $g(x)=f(ax)$ for all x. Then

$$f\left(a.\frac{2\pi n}{a}\right) = g\left(\frac{2\pi n}{a}\right),$$

and we define the Fourier transform of $f(x)$ without the factor $\frac{1}{\sqrt{2\pi}}$ so that

$$\begin{aligned} F(n) &= \int_{-\infty}^{\infty} e^{-inx} f(x)\,dx = \int_{-\infty}^{\infty} e^{-inx} f\left(a.\frac{x}{a}\right)\,dx \\ &= \int_{-\infty}^{\infty} e^{-inx} g\left(\frac{x}{a}\right)\,dx \\ &= a\int_{-\infty}^{\infty} e^{-i(an)y}\, g(y)\,dy \\ &= a\,G(an). \end{aligned}$$

Consequently, equality (2.6.13) reduces to

$$\sum_{n=-\infty}^{\infty} g(\frac{2\pi n}{a}) = \frac{a}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} G(an). \tag{2.6.15}$$

Putting $b=\frac{2\pi}{a}$ in (2.6.15) gives

$$\sum_{n=-\infty}^{\infty} g(bn) = \sqrt{2\pi}\, b^{-1} \sum_{n=-\infty}^{\infty} G(2\pi b^{-1} n). \tag{2.6.16}$$

When $b = 2\pi$, result (2.6.16) becomes (2.6.13). We apply these formulas to prove the following series

$$\text{(a)} \qquad \sum_{n=-\infty}^{\infty} \frac{1}{(n^2+b^2)} = \frac{\pi}{b}\coth(\pi b), \tag{2.6.17}$$

$$\text{(b)} \qquad \sum_{n=-\infty}^{\infty} \exp(-\pi n^2 t) = \frac{1}{\sqrt{t}} \sum_{n=-\infty}^{\infty} \exp\left(-\frac{\pi n^2}{t}\right), \tag{2.6.18}$$

$$\text{(c)} \qquad \sum_{n=-\infty}^{\infty} \frac{1}{(x+n\pi)^2} = \operatorname{cosec}^2(x). \tag{2.6.19}$$

To prove (a), we write $f(x) = (x^2+b^2)^{-1}$ so that $F(k) = \sqrt{\frac{\pi}{2}}\,\frac{1}{b}\exp(-b|k|)$. We now use (2.6.14) to derive

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n^2+b^2)} = \frac{\pi}{b} \sum_{n=-\infty}^{\infty} \exp(-2|n|\pi b)$$
$$= \frac{\pi}{b}\left[\sum_{n=0}^{\infty} \exp(-2n\pi b) + \sum_{n=1}^{\infty} \exp(2n\pi b)\right]$$

which is, by writing $r = \exp(-2\pi b)$,

$$= \frac{\pi}{b}\left[\sum_{n=0}^{\infty} r^n + \sum_{n=1}^{\infty}\left(\frac{1}{r}\right)^n\right] = \frac{\pi}{b}\left(\frac{r}{1-r} + \frac{1}{1-r}\right)$$
$$= \frac{\pi}{b}\left(\frac{1+r}{1-r}\right) = \frac{\pi}{b}\coth(\pi b).$$

It follows from (2.6.14) that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n^2+b^2)} = \frac{\pi}{b}\frac{\left(1+e^{-2\pi b}\right)}{\left(1-e^{-2\pi b}\right)}.$$

Or,

$$2\sum_{n=1}^{\infty} \frac{1}{(n^2+b^2)} + \frac{1}{b^2} = \frac{\pi}{b}\frac{\left(1+e^{-2\pi b}\right)}{\left(1-e^{-2\pi b}\right)}.$$

It turns out that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{(n^2+b^2)} &= \frac{\pi}{2b}\left[\frac{\left(1+e^{-2\pi b}\right)}{\left(1-e^{-2\pi b}\right)}-\frac{1}{\pi b}\right] \\ &= \frac{\pi^2}{x}\left[\frac{(1+e^{-x})}{(1-e^{-x})}-\frac{2}{x}\right], \qquad (2\pi b = x) \\ &= \frac{\pi^2}{x^2}\left[\frac{x\,(1+e^{-x})-2\,(1-e^{-x})}{(1-e^{-x})}\right] \\ &= \left(\frac{\pi}{x}\right)^2\left[\frac{x^3\left(\frac{1}{2}-\frac{1}{3}\right)-\frac{x^4}{12}+\ldots.}{x-\frac{x^2}{2!}+\frac{x^3}{3!}-\ldots.}\right].\end{aligned}$$

In the limit as $b \to 0\ (x \to 0)$, we obtain the well-known Euler's result

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}. \tag{2.6.20}$$

To prove (b), we assume $f(x)=\exp(-\pi t x^2)$ so that $F(k)=\frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{k^2}{4\pi t}\right)$. Thus, the Poisson formula (2.6.14) gives

$$\sum_{n=-\infty}^{\infty}\exp(-\pi t n^2)=\frac{1}{\sqrt{t}}\sum_{n=-\infty}^{\infty}\exp(-\pi n^2/t).$$

This identity plays an important role in number theory and in the theory of elliptic functions. The *Jacobi theta function* $\Theta(s)$ is defined by

$$\Theta(s)=\sum_{n=-\infty}^{\infty}\exp(-\pi s n^2), \qquad s>0, \tag{2.6.21}$$

so that (2.6.16) gives the *functional equation* for the theta function

$$\sqrt{s}\,\Theta(s)=\Theta\left(\frac{1}{s}\right). \tag{2.6.22}$$

The theta function $\Theta(s)$ also extends to complex values of s when $Re(s)>0$ and the functional equation is still valid for complex s. The theta function is closely related to the *Riemann zeta function* $\zeta(s)$ defined for $Re(s)>1$ by

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}. \tag{2.6.23}$$

An integral representation of $\zeta(s)$ can be found from the result

$$\int_0^{\infty} x^{s-1}\,e^{-nx}dx=\frac{\Gamma(s)}{n^s}, \qquad Re(s)>0,$$

where the *gamma function* $\Gamma(s)$ is defined by

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt, \qquad Re(s) > 0.$$

Summing both sides of this result and interchanging the order of summation and integration, which is permissible for $Re(s) > 1$, gives

$$\Gamma(s)\,\zeta(s) = \int_0^\infty x^{s-1} \frac{dx}{e^x - 1}, \qquad Re(s) > 1. \tag{2.6.24}$$

It turns out that $\zeta(s)$, $\Theta(s)$, and $\Gamma(s)$ are related by the following identity:

$$\zeta(s)\Gamma(s/2) = \frac{1}{2}\pi^{s/2} \int_0^\infty x^{s/2-1} \left[\Theta(x) - 1\right] dx, \qquad Re(s) > 1. \tag{2.6.25}$$

Considering the complex integral in a suitable closed contour C

$$I = \frac{1}{2\pi i} \int_C \frac{z^{s-1}}{e^{-z} - 1} dz,$$

and using the Cauchy residue theorem with all zeros of $(e^{-z} - 1)$ at $z = 2\pi i n$, $n = \pm 1, \pm 2, ..., \pm N$ gives

$$I = -2\sin\left(\frac{\pi s}{2}\right) \sum_{n=1}^{\infty} (2\pi n)^{s-1}.$$

To prove (c), we use the Fourier transform of the function $f(x) = (1 - |x|)$ $H(1 - |x|)$ to obtain the result. In the limit as $N \to \infty$, the sum of the residues is convergent so that the integral gives the relation

$$2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \zeta(1-s) = \frac{\zeta(s)}{\Gamma(1-s)}. \tag{2.6.26}$$

In view of another relation for the gamma function, $\Gamma(1+z)\Gamma(-z) = -\frac{\pi}{\sin \pi z}$, the relation (2.6.26) leads to a famous functional relation for $\zeta(s)$ in the form

$$\pi^s \zeta(1-s) = 2^{1-s} \Gamma(s) \cos\left(\frac{\pi s}{2}\right) \zeta(s). \tag{2.6.27}$$

2.7 The Shannon Sampling Theorem

An analog signal $f(t)$ is a continuous function of time t defined in $-\infty < t < \infty$, with the exception of perhaps a countable number of jump discontinuities.

Almost all analog signals $f(t)$ of interest in engineering have finite energy. By this we mean that $f \in L^2(-\infty, \infty)$. The norm of f defined by

$$||f|| = \left[\int_{-\infty}^{\infty} |f(t)|^2 \, dt\right]^{\frac{1}{2}} \tag{2.7.1}$$

represents the square root of the total energy content of the signal $f(t)$. The *spectrum* of a signal $f(t)$ is represented by its Fourier transform $F(\omega)$, where ω is called the *frequency*. The frequency is measured by $\nu = \frac{\omega}{2\pi}$ in terms of Hertz.

A continuous signal $f_a(t)$ is called *band limited* if its Fourier transform $F(\omega)$ is zero except in a finite interval $-a \leq t \leq a$, that is, if

$$F_a(\omega) = 0 \qquad \text{for} \quad |\omega| > a. \tag{2.7.2}$$

Then $a(>0)$ is called the *cutoff frequency.*

In particular, if

$$F(\omega) = \begin{Bmatrix} 1, & |\omega| \leq a \\ 0, & |\omega| > a \end{Bmatrix} \tag{2.7.3}$$

then $F(\omega)$ is called a *gate function* and is denoted by $F_a(\omega)$, and the band limited signal is denoted by $f_a(t)$. If a is the smallest value for which (2.7.2) holds, it is called the *bandwidth* of the signal. Even if an analog signal $f(t)$ is not band-limited, we can reduce it to a band-limited signal by what is called an *ideal low-pass filtering.* To reduce $f(t)$ to a band-limited signal $f_a(t)$ with bandwidth less than or equal to a, we consider

$$F_a(\omega) = \begin{Bmatrix} F(\omega), & |\omega| \leq a \\ 0, & |\omega| > a \end{Bmatrix} \tag{2.7.4}$$

and find the low-pass filter function $f_a(t)$ by the inverse Fourier transform

$$f_a(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t} F_a(\omega) d\omega = \frac{1}{2\pi}\int_{-a}^{a} e^{i\omega t} F_a(\omega) d\omega. \tag{2.7.5}$$

This function $f_a(t)$ is called the *Shannon sampling function.* When $a = \pi$, $f_\pi(t)$ is called the *Shannon scaling function.* The band-limited signal $f_a(t)$ is given by

$$f_a(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega = \frac{1}{2\pi}\int_{-a}^{a} e^{i\omega t} d\omega = \frac{\sin at}{\pi t}. \tag{2.7.6}$$

Both $F(\omega)$ and $f_a(t)$ are shown in Figure 2.6 for $a = 2$.

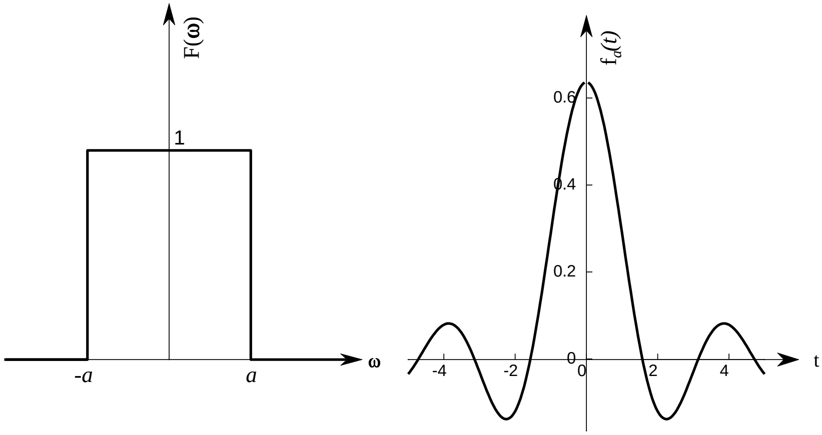

Figure 2.6 The gate function and its inverse Fourier transform.

Consider the limit as $a \to \infty$ of the Fourier integral for $-\infty < \omega < \infty$

$$1 = \lim_{a\to\infty} \int_{-\infty}^{\infty} e^{-i\omega t} f_a(t)dt = \lim_{a\to\infty} \int_{-\infty}^{\infty} e^{-i\omega t} \frac{\sin at}{\pi t} dt$$
$$= \int_{-\infty}^{\infty} e^{-i\omega t} \left[\lim_{a\to\infty} \frac{\sin at}{\pi t}\right] dt = \int_{-\infty}^{\infty} e^{-i\omega t} \delta(t) dt.$$

Clearly, the delta function $\delta(t)$ can be thought of as the limit of the sequence of functions $f_a(t)$. More precisely,

$$\delta(t) = \lim_{a\to\infty} \left(\frac{\sin at}{\pi t}\right). \tag{2.7.7}$$

We next consider the band-limited signal

$$f_a(t) = \frac{1}{2\pi} \int_{-a}^{a} F(\omega) e^{i\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, F_a(\omega)\, e^{i\omega t} d\omega,$$

which is, by the Convolution Theorem,

$$f_a(t) = \int_{-\infty}^{\infty} f(\tau) f_a(t-\tau) d\tau = \int_{-\infty}^{\infty} \frac{\sin a(t-\tau)}{\pi(t-\tau)} f(\tau) d\tau. \tag{2.7.8}$$

This integral represents the *sampling integral representation* of the band-limited signal $f_a(t)$.

Example 2.7.1 (*Synthesis and Resolution of a Signal; Physical Interpretation of Convolution*).
In electrical engineering problems, a time-dependent electric, optical or electromagnetic *pulse* is usually called a *signal.* Such a signal can be considered as a superposition of plane waves of all real frequencies so that it can be represented by the inverse Fourier transform

$$f(t)=\mathscr{F}^{-1}\{F(\omega)\}=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)e^{i\omega t}d\omega, \tag{2.7.9}$$

where $F(\omega)=\mathscr{F}\{f(t)\}$, the factor $(1/2\pi)$ is introduced because the angular frequency ω is related to linear frequency ν by $\omega=2\pi\nu$, and negative frequencies are introduced for mathematical convenience so that we can avoid dealing with the cosine and sine functions separately. Clearly, $F(\omega)$ can be represented by the Fourier transform of the signal $f(t)$ as

$$F(\omega)=\int_{-\infty}^{\infty}f(t)e^{-i\omega t}dt. \tag{2.7.10}$$

This represents the *resolution* of the signal into its angular frequency components, and (2.7.9) gives a *synthesis* of the signal from its individual components.

Consider a simple electrical device such as an amplifier with an input signal $f(t)$, and an output signal $g(t)$. For an input of a single frequency ω, $f(t)=e^{i\omega t}$. The amplifer will change the amplitude and may also change the phase so that the output can be expressed in terms of the input, the amplitude and the phase modifying function $\Phi(\omega)$ as

$$g(t)=\Phi(\omega)f(t), \tag{2.7.11}$$

where $\Phi(\omega)$ is usually known as the *transfer function* and is, in general, a complex function of the real variable ω. This function is generally independent of the presence or absence of any other frequency components. Thus, the total output may be found by integrating over the entire input as modified by the amplifier

$$g(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi(\omega)F(\omega)\,e^{i\omega t}d\omega. \tag{2.7.12}$$

Thus, the total output signal can readily be calculated from any given input signal $f(t)$. On the other hand, the transfer function $\Phi(\omega)$ is obviously characteristic of the amplifier device and can, in general, be obtained as the Fourier transform of some function $\phi(t)$ so that

$$\Phi(\omega)=\int_{-\infty}^{\infty}\phi(t)e^{-i\omega t}dt. \tag{2.7.13}$$

The Convolution Theorem 2.5.5 allows us to rewrite (2.7.12) as

$$g(t) = \mathscr{F}^{-1}\{\Phi(\omega)F(\omega)\} = f(t) * \phi(t) = \int_{-\infty}^{\infty} f(\tau)\phi(t-\tau)d\tau. \tag{2.7.14}$$

Physically, this result represents an output signal $g(t)$ as the integral superposition of an input signal $f(t)$ modified by $\phi(t-\tau)$. Linear translation invariant systems, such as *sensors* and *filters*, are modeled by the convolution equations $g(t) = f(t) * \phi(t)$, where $\phi(t)$ is the system impulse response function. In fact (2.7.14) is the most general mathematical representation of an output (effect) function in terms of an input (cause) function modified by the amplifier where t is the time variable. Assuming the principle of causality, that is, every effect has a cause, we must require $\tau < t$. The principle of causality is imposed by requiring

$$\phi(t-\tau) = 0 \qquad \text{when } \tau > t. \tag{2.7.15}$$

Consequently, (2.7.14) gives

$$g(t) = \int_{-\infty}^{t} f(\tau)\phi(t-\tau)d\tau. \tag{2.7.16}$$

In order to determine the significance of $\phi(t)$, we use an impulse function $f(\tau) = \delta(\tau)$ so that (2.7.16) becomes

$$g(t) = \int_{-\infty}^{t} \delta(\tau)\phi(t-\tau)d\tau = \phi(t)H(t). \tag{2.7.17}$$

This recognizes $\phi(t)$ as the output corresponding to a unit impulse at $t = 0$, and the Fourier transform of $\phi(t)$ is

$$\Phi(\omega) = \mathscr{F}\{\phi(t)\} = \int_{0}^{\infty} \phi(t)e^{-i\omega t}dt, \tag{2.7.18}$$

with $\phi(t) = 0$ for $t < 0$. ▯

Example 2.7.2 (*The Series Sampling Expansion of a Band-Limited Signal*). Consider a band-limited signal $f_a(t)$ with Fourier transform $F(\omega) = 0$ for $|\omega| > a$. We write the Fourier series expansion of $F(\omega)$ on the interval $-a < \omega < a$ in terms of the orthogonal set of functions $\left\{\exp\left(-\frac{in\pi\omega}{a}\right)\right\}$ in the form

$$F(\omega) = \sum_{n=-\infty}^{\infty} a_n \exp\left(-\frac{in\pi}{a}\omega\right), \tag{2.7.19}$$

where the Fourier coefficients a_n are given by

$$a_n = \frac{1}{2a}\int_{-a}^{a} F(\omega)\exp\left(\frac{in\pi}{a}\omega\right)d\omega = \frac{1}{2a}\ f_a\left(\frac{n\pi}{a}\right). \tag{2.7.20}$$

Thus, the Fourier series expansion (2.7.19) becomes

$$F(\omega) = \frac{1}{2a}\sum_{n=-\infty}^{\infty} f_a\left(\frac{n\pi}{a}\right)\exp\left(-\frac{in\pi}{a}\omega\right). \tag{2.7.21}$$

The signal function $f_a(t)$ is obtained by multiplying (2.7.21) by $e^{i\omega t}$ and integrating over $(-a,a)$ so that

$$\begin{aligned}
f_a(t) &= \int_{-a}^{a} F(\omega)e^{i\omega t}d\omega \\
&= \frac{1}{2a}\int_{-a}^{a} e^{i\omega t}d\omega\left[\sum_{n=-\infty}^{\infty} f_a\left(\frac{n\pi}{a}\right)\exp\left(-\frac{in\pi}{a}\omega\right)\right] \\
&= \frac{1}{2a}\sum_{n=-\infty}^{\infty} f_a\left(\frac{n\pi}{a}\right)\int_{-a}^{a}\exp\left[i\omega\left(t-\frac{n\pi}{a}\right)\right]d\omega \\
&= \sum_{n=-\infty}^{\infty} f_a\left(\frac{n\pi}{a}\right)\frac{\sin a\left(t-\frac{n\pi}{a}\right)}{a\left(t-\frac{n\pi}{a}\right)} \\
&= \sum_{n=-\infty}^{\infty} f_a\left(\frac{n\pi}{a}\right)\frac{\sin\left(at-n\pi\right)}{\left(at-n\pi\right)}.
\end{aligned} \tag{2.7.22}$$

This result is the main content of the sampling theorem. It simply states that a band-limited signal $f_a(t)$ can be reconstructed from the infinite set of discrete samples of $f_a(t)$ at $t=0,\ \pm\frac{\pi}{a},\ \ldots.$. In practice, a discrete set of samples is useful in the sense that most systems receive discrete samples $\{f(t_n)\}$ as an input. The sampling theorem can be realized physically. Modern telephone equipment employs sampling to send messages over wires. In fact, it seems that sampling is audible on some transoceanic cable calls.

Result (2.7.22) can be obtained from the convolution theorem by using discrete input samples

$$\sum_{n=-\infty}^{\infty}\frac{\pi}{a}\,f_a\left(\frac{n\pi}{a}\right)\delta\left(t-\frac{n\pi}{a}\right) = f(t). \tag{2.7.23}$$

Hence, the sampling expansion (2.7.8) gives the band-limited signal

$$f_a(t) = \int_{-\infty}^{\infty} \frac{\sin a(t-\tau)}{\pi(t-\tau)} \left[\sum_{n=-\infty}^{\infty} \frac{\pi}{a} f_a\left(\frac{n\pi}{a}\right) \delta\left(\tau - \frac{n\pi}{a}\right)\right] d\tau$$

$$= \sum_{n=-\infty}^{\infty} f_a\left(\frac{n\pi}{a}\right) \int_{-\infty}^{\infty} \frac{\sin a(t-\tau)}{a(t-\tau)} \delta\left(\tau - \frac{n\pi}{a}\right) d\tau$$

$$= \sum_{n=-\infty}^{\infty} f_a\left(\frac{n\pi}{a}\right) \frac{\sin a\left(t - \frac{n\pi}{a}\right)}{a\left(t - \frac{n\pi}{a}\right)}. \tag{2.7.24}$$

□

In general, the output can be best described by taking the Fourier transform of (2.7.14) so that

$$G(\omega) = F(\omega)\Phi(\omega), \tag{2.7.25}$$

where $\Phi(\omega)$ is called the *transfer function* of the system. Thus, the output can be calculated from (2.7.25) by the Fourier inversion formula

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, \Phi(\omega)\, e^{i\omega t}\, d\omega. \tag{2.7.26}$$

Obviously, the transfer function $\Phi(\omega)$ is a characteristic of a linear system. A linear system is a *filter* if it possesses signals of certain frequencies and attenuates others. If the transfer function

$$\Phi(\omega) = 0 \qquad |\omega| \geq \omega_0, \tag{2.7.27}$$

then $\phi(t)$, the Fourier inverse of $\Phi(\omega)$, is called a *low-pass filter.*

On the other hand, if the transfer function

$$\Phi(\omega) = 0 \qquad |\omega| \leq \omega_1, \tag{2.7.28}$$

then $\phi(t)$ is a *high-pass filter.* A *band-pass filter* possesses a band $\omega_0 \leq |\omega| \leq \omega_1$. It is often convenient to express the system transfer function $\Phi(\omega)$ in the complex form

$$\Phi(\omega) = A(\omega)\exp[-i\theta(\omega)], \tag{2.7.29}$$

where $A(\omega)$ is called the *amplitude* and $\theta(\omega)$ is called the *phase* of the transfer function. Obviously, the system impulse response $\phi(t)$ is given by the inverse Fourier transform

$$\phi(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} A(\omega) \exp[i\{\omega t - \theta(\omega)\}]\, d\omega. \tag{2.7.30}$$

For a unit step function as the input $f(t)=H(t)$, we have

$$F(\omega)=\hat{H}(\omega)=\left(\pi\delta(\omega)+\frac{1}{i\omega}\right),$$

where $\hat{H}(\omega)=\mathscr{F}\{H(t)\}$ and the associated output $g(t)$ is then given by

$$\begin{aligned} g(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi(\omega)\hat{H}(\omega)e^{i\omega t}\,d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\pi\delta(\omega)+\frac{1}{i\omega}\right)A(\omega)\exp[i\{\omega t-\theta(\omega)\}]\,d\omega \\ &= \frac{1}{2}A(0)+\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{A(\omega)}{\omega}\exp\left[i\left\{\omega t-\theta(\omega)-\frac{\pi}{2}\right\}\right]\,d\omega\,. \end{aligned} \tag{2.7.31}$$

We next give another characterization of a filter in terms of the amplitude of the transfer function.

A filter is called *distortionless* if its output $g(t)$ to an arbitrary input $f(t)$ has the same form as the input, that is,

$$g(t)=A_0 f(t-t_0). \tag{2.7.32}$$

Evidently,

$$G(\omega)=A_0 e^{-i\omega t_0}\,F(\omega)=\Phi(\omega)\,F(\omega)$$

where

$$\Phi(\omega)=A_0 e^{-i\omega t_0}$$

represents the transfer function of the distortionless filter. It has a constant amplitude A_0 and a linear phase shift $\theta(\omega)=\omega t_0$.

However, in general, the amplitude $A(\omega)$ of a transfer function is not constant, and the phase $\theta(\omega)$ is not a linear function.

A filter with constant amplitude, $|\theta(\omega)|=A_0$ is called an *all-pass filter.* It follows from Parseval's formula that the energy of the output of such a filter is proportional to the energy of its input.

A filter whose amplitude is constant for $|\omega|<\omega_0$ and zero for $|\omega|>\omega_0$ is called an *ideal low-pass filter.* More explicitly, the amplitude is given by

$$A(\omega)=A_0\hat{H}(\omega_0-|\omega|)=A_0\hat{\chi}_{\omega_0}(\omega)\,, \tag{2.7.33}$$

where $\hat{\chi}_{\omega_0}(\omega)$ is a rectangular pulse. So, the transfer function of the low-pass filter is

$$\Phi(\omega)=A_0\hat{\chi}_{\omega_0}(\omega)\exp(-i\omega t_0)\,. \tag{2.7.34}$$

Finally, the *ideal high-pass filter* is characterized by its amplitude given by

$$A(\omega)=A_0\hat{H}(|\omega|-\omega_0)=A_0\hat{\chi}_{\omega_0}(\omega)\,, \tag{2.7.35}$$

where A_0 is a constant. Its transfer function is given by

$$\Phi(\omega) = A_0\,[1 - \hat{\chi}_{\omega_0}(\omega)]\exp(-i\omega t_0)\,. \tag{2.7.36}$$

Example 2.7.3 (*Bandwidth and Bandwidth Equation*).
The Fourier spectrum of a signal (or waveform) gives an indication of the frequencies that exist during the total duration of the signal (or *waveform*). From the knowledge of the frequencies that are present, we can calculate the average frequency and the spread about that average. In particular, if the signal is represented by $f(t)$, we can define its Fourier spectrum by

$$F(\nu) = \int_{-\infty}^{\infty} e^{-2\pi i\nu t}\, f(t)\, dt. \tag{2.7.37}$$

Using $|F(\nu)|^2$ for the density in frequency, the average frequency is denoted by $<\nu>$ and defined by

$$<\nu> = \int_{-\infty}^{\infty} \nu\, |F(\nu)|^2\, d\nu. \tag{2.7.38}$$

The bandwidth is then the *root mean square* (RMS) deviation at about the average, that is,

$$B^2 = \int_{-\infty}^{\infty} (\nu - <\nu>)^2\, d\nu. \tag{2.7.39}$$

Expressing the signal in terms of its amplitude and phase

$$f(t) = a(t)\, \exp\{i\theta t\}, \tag{2.7.40}$$

the instantaneous frequency, $\nu(t)$ is the frequency at a particular time defined by

$$\nu(t) = \frac{1}{2\pi}\,\theta'(t). \tag{2.7.41}$$

Substituting (2.7.37) and (2.7.40) into (2.7.38) gives

$$<\nu> = \frac{1}{2\pi}\int_{-\infty}^{\infty} \theta'(t)\, a^2(t)\, dt = \int_{-\infty}^{\infty} \nu(t)\, a^2(t)\, dt. \tag{2.7.42}$$

This formula states that the average frequency is the average value of the instantaneous frequency weighted by the square of the amplitude of the signal.

We next derive the bandwidth equation in terms of the amplitude and phase of the signal in the form

$$B^2 = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} \left[\frac{a'(t)}{a(t)}\right]^2 a^2(t)\, dt + \int_{-\infty}^{\infty} \left[\frac{1}{2\pi}\,\theta'(t) - <\nu>\right]^2 a^2(t)\, dt. \tag{2.7.43}$$

A straightforward but lengthy way to derive it is to substitute (2.7.40) into (2.7.39) and simplify. However, we give an elegant derivation of (2.7.43) by representing the frequency by the operator

$$\nu = \frac{1}{2\pi i}\,\frac{d}{dt}. \tag{2.7.44}$$

We calculate the average by sandwiching the operator between the complex conjugate of the signal and the signal. Thus,

$$\begin{aligned}
<\nu> &= \int_{-\infty}^{\infty} \nu\,|F(\nu)|^2\,d\nu = \int_{-\infty}^{\infty} \bar{f}(t)\left[\frac{1}{2\pi i}\,\frac{d}{dt}\right] f(t)\,dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} a(t)\,\{-ia'(t) + a(t)\theta'(t)\}\,dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} -\frac{1}{2}i\left[\frac{d}{dt}a^2(t)\right]\,dt + \frac{1}{2\pi}\int_{-\infty}^{\infty} a^2(t)\theta'(t)\,dt \qquad (2.7.45)\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \theta'(t)a^2(t)\,dt \qquad (2.7.46)
\end{aligned}$$

provided the first integral in (2.7.44) vanishes if $a(t)\to 0$ as $|t|\to\infty$.

It follows from the definition (2.7.39) of the bandwidth that

$$\begin{aligned}
B^2 &= \int_{-\infty}^{\infty} (\nu - <\nu>)^2\,|F(\nu)|^2\,d\nu \\
&= \int_{-\infty}^{\infty} \bar{f}(t)\left[\frac{1}{2\pi i}\frac{d}{dt} - <\nu>\right]^2 f(t)\,dt \\
&= \int_{-\infty}^{\infty} \left|\left[\frac{1}{2\pi i}\frac{d}{dt} - <\nu>\right] f(t)\right|^2\,dt \\
&= \int_{-\infty}^{\infty} \left|\frac{1}{2\pi i}\frac{a'(t)}{a(t)} + \frac{1}{2\pi}\theta'(t) - <\nu>\right|^2\,a^2(t)\,dt \\
&= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\left[\frac{a'(t)}{a(t)}\right]^2 a^2(t)\,dt + \int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\,\theta'(t) - <\nu>\right]^2 a^2(t)\,dt.
\end{aligned}$$

This completes the derivation. ▯

Physically, the second term in equation (2.7.43) gives averages of all of the deviations of the instantaneous frequency from the average frequency. In electrical engineering literature, the spread of frequency about the *instantaneous frequency*, which is defined as an average of the frequencies that exist at a particular time, is called *instantaneous bandwidth*, given by

$$\sigma^2_{\nu/t} = \frac{1}{(2\pi)^2}\left[\frac{a'(t)}{a(t)}\right]^2. \tag{2.7.47}$$

In the case of a chirp with a Gaussian envelope

$$f(t)=\left(\frac{\alpha}{\pi}\right)^{\frac{1}{4}}\exp\left[-\frac{1}{2}\alpha t^2+\frac{1}{2}i\beta\alpha t^2+2\pi i\nu_0 t\right], \tag{2.7.48}$$

where its *Fourier spectrum* is given by

$$F(\nu)=(\alpha\pi)^{\frac{1}{4}}\left(\frac{1}{\alpha-i\beta}\right)^{\frac{1}{2}}\exp\left[-2\pi^2(\nu-\nu_0)^2/(\alpha-i\beta)\right]. \tag{2.7.49}$$

The *energy density spectrum* of the signal is

$$|F(\nu)|^2=2\left(\frac{\alpha\pi}{\alpha^2+\beta^2}\right)^{\frac{1}{2}}\exp\left[-\frac{4\alpha\pi^2(\nu-\nu_0)^2}{\alpha^2+\beta^2}\right]. \tag{2.7.50}$$

Finally, the average frequency $<\nu>$ and the bandwidth square are respectively given by

$$<\nu>=\nu_0 \quad \text{and} \quad B^2=\frac{1}{8\pi^2}\left(\alpha+\frac{\beta^2}{\alpha}\right). \tag{2.7.51}$$

A large bandwidth can be achieved in two very qualitatively different ways. The amplitude modulation can be made large by taking α large, and the frequency modulation can be small by letting $\beta\to 0$. It is possible to make the frequency modulation large by making β large and α very small. These two extreme situations are physically very different even though they produce the same bandwidth.

Example 2.7.4 Find the transfer function and the corresponding *impulse response function* of the RLC circuit governed by the differential equation

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{1}{C}q=e(t) \tag{2.7.52}$$

where $q(t)$ is the charge, R, L, C are constants, and $e(t)$ is the given voltage (input).

Equation (2.7.25) provides the definition of the *transfer function* in the frequency domain

$$\Phi(\omega)=\frac{G(\omega)}{F(\omega)}=\frac{\mathscr{F}\{g(t)\}}{\mathscr{F}\{f(t)\}}, \tag{2.7.53}$$

where $\phi(t)=\mathscr{F}^{-1}\{\Phi(\omega)\}$ is called the *impulse response function*.

Taking the Fourier transform of (2.7.52) gives

$$\left(-L\omega^2+Ri\omega+\frac{1}{C}\right)Q(\omega)=E(\omega). \tag{2.7.54}$$

Thus, the transfer function is

$$\begin{aligned}\Phi(\omega) &= \frac{Q(\omega)}{E(\omega)} = \frac{-C}{LC\omega^2 - iRC\omega - 1} \\ &= \frac{i}{2L\beta}\left[\frac{1}{\omega - i(\alpha+\beta)} - \frac{1}{\omega - i(\alpha-\beta)}\right], \end{aligned} \tag{2.7.55}$$

where

$$\alpha = \frac{R}{2L} \quad \text{and} \quad \beta = \left[\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}\right]^{\frac{1}{2}}. \tag{2.7.56}$$

The inverse Fourier transform of (2.7.55) yields the impulse response function

$$\phi(t) = \frac{1}{2\beta L}\left(e^{\beta t} - e^{-\beta t}\right)e^{-\alpha t}H(t). \tag{2.7.57}$$

$\square$

2.8 The Gibbs Phenomenon

We now examine the so-called the *Gibbs jump phenomenon* which deals with the limiting behavior of a band-limited signal $f_{\omega_0}(t)$ represented by the sampling integral representation (2.7.8) at a point of discontinuity of $f(t)$. This phenomenon reveals the intrinsic overshoot near a jump discontinuity of a function associated with the Fourier series. More precisely, the partial sums of the Fourier series overshoot the function near the discontinuity, and the overshoot continues no matter how many terms are taken in the partial sum. However, the Gibbs phenomenon does not occur if the partial sums are replaced by the Cesaro means, the average of the partial sums.

In order to demonstrate the Gibbs phenomenon, we rewrite (2.7.8) in the convolution form

$$f_{\omega_0}(t) = \int_{-\infty}^{\infty} f(\tau)\,\frac{\sin\omega_0(t-\tau)}{\pi(t-\tau)}\,d\tau = (f * \delta_{\omega_0})(t), \tag{2.8.1}$$

where

$$\delta_{\omega_0}(t) = \frac{\sin\omega_0 t}{\pi t}. \tag{2.8.2}$$

Clearly, at every point of continuity of $f(t)$, we have

$$\begin{aligned}\lim_{\omega_0\to\infty} f_{\omega_0}(t) &= \lim_{\omega_0\to\infty}\left(f*\delta_{\omega_0}\right)(t) = \lim_{\omega_0\to\infty}\int_{-\infty}^{\infty} f(\tau)\,\frac{\sin\omega_0(t-\tau)}{\pi(t-\tau)}\,d\tau \\ &= \int_{-\infty}^{\infty} f(\tau)\left[\lim_{\omega_0\to\infty}\frac{\sin\omega_0(t-\tau)}{\pi(t-\tau)}\right]d\tau \\ &= \int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)\,d\tau = f(t)\,. \end{aligned} \tag{2.8.3}$$

We now consider the limiting behavior of $f_{\omega_0}(t)$ at the point of discontinuity $t=t_0$. To simplify the calculation, we set $t_0=0$ so that we can write $f(t)$ as a sum of a continuous function, $f_c(t)$ and a suitable step function

$$f(t) = f_c(t) + [f(0+) - f(0-)]\; H(t). \tag{2.8.4}$$

Replacing $f(t)$ by the right-hand side of (2.8.4) in Equation (2.8.1) yields

$$\begin{aligned} f_{\omega_0}(t) &= \int_{-\infty}^{\infty} f_c(\tau)\,\frac{\sin\omega_0(t-\tau)}{\pi(t-\tau)}\,d\tau \\ &\quad + [f(0+) - f(0-)]\int_{-\infty}^{\infty} H(\tau)\,\frac{\sin\omega_0(t-\tau)}{\pi(t-\tau)}\,d\tau \\ &= f_c(t) + [f(0+) - f(0-)]\; H_{\omega_0}(t)\,, \end{aligned} \tag{2.8.5}$$

where

$$\begin{aligned} H_{\omega_0}(t) &= \int_{-\infty}^{\infty} H(\tau)\,\frac{\sin\omega_0(t-\tau)}{\pi(t-\tau)}\,d\tau = \int_{0}^{\infty} \frac{\sin\omega_0(t-\tau)}{\pi(t-\tau)}\,d\tau \\ &= \int_{-\infty}^{\omega_0 t} \frac{\sin x}{\pi x}\,dx \qquad (\text{putting } \omega_0(t-\tau)=x) \\ &= \left(\int_{-\infty}^{0} + \int_{0}^{\omega_0 t}\right)\left(\frac{\sin x}{\pi x}\right)dx = \left(\int_{0}^{\infty} + \int_{0}^{\omega_0 t}\right)\left(\frac{\sin x}{\pi x}\right)dx \\ &= \frac{1}{2} + \frac{1}{\pi}\,si(\omega_0 t)\,, \end{aligned} \tag{2.8.6}$$

and the function $si(t)$ is defined by

$$si(t) = \int_0^t \frac{\sin x}{x}\,dx\,. \tag{2.8.7}$$

Note that

$$H_{\omega_0}\left(\frac{\pi}{\omega_0}\right) = \frac{1}{2} + \int_0^{\pi}\frac{\sin x}{\pi x}\,dx > 1\,, \quad H_{\omega_0}\left(-\frac{\pi}{\omega_0}\right) = \frac{1}{2} - \int_0^{\pi}\frac{\sin x}{\pi x}\,dx < 0\,.$$

Clearly, for a fixed ω_0, $\frac{1}{\pi}si(\omega_0 t)$ attains its maximum at $t=\frac{\pi}{\omega_0}$ in $(0,\infty)$ and minimum at $t=-\frac{\pi}{\omega_0}$, since for a larger t the integrand oscillates with decreasing amplitudes. The function $H_{\omega_0}(t)$ is shown in Figure 2.7 since $H_{\omega_0}(0)=\frac{1}{2}$

and $f_c(0) = f(0-)$ and

$$f_{\omega_0}(0) = f_c(0) + \frac{1}{2}\left[f(0+) - f(0-)\right] = \frac{1}{2}\left[f(0+) + f(0-)\right].$$

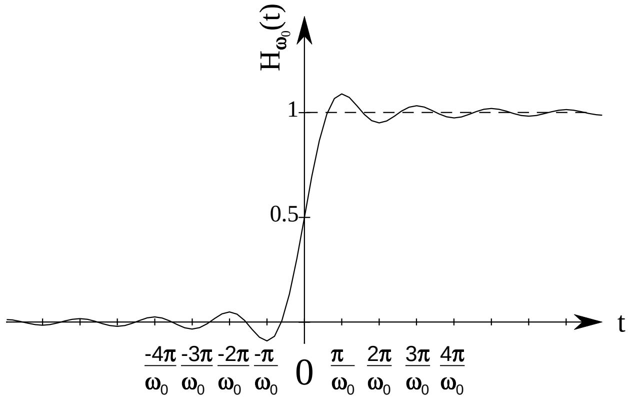

Figure 2.7 Graph of $H_{\omega_0}(t)$.

Thus, the graph of $H_{\omega_0}(t)$ shows that as ω_0 increases, the time scale changes, and the ripples remain the same. In the limit $\omega_0 \to \infty$, the convergence of $H_{\omega_0}(t) = (H * \delta_{\omega_0})(t)$ to $H(t)$ exhibits the intrinsic overshoot leading to the classical Gibbs phenomenon.

Example 2.8.1 (*The Square Wave Function and the Gibbs Phenomenon*). Consider the single-pulse square function defined by

$$f(x) = \left\{ \begin{array}{ll} 1, & -a < x < a \\ \frac{1}{2}, & x = \pm a \\ 0, & |x| > a \end{array} \right\}.$$

The graph of $f(x)$ is given in Figure 2.8.

Thus,

$$F(k) = \mathscr{F}\{f(x)\} = \sqrt{\frac{2}{\pi}}\left(\frac{\sin ak}{k}\right).$$

▯

We next define a function $f_\lambda(x)$ by the integral

$$f_\lambda(x) = \int_{-\lambda}^{\lambda} F(k)\, e^{ikx}\, dk.$$

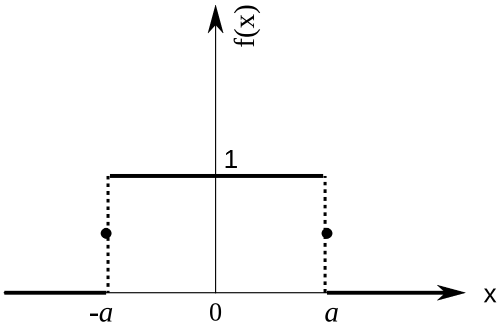

Figure 2.8 The square wave function.

As $|\lambda| \to \infty$, $f_\lambda(x)$ will tend pointwise to $f(x)$ for all x. Convergence occurs even at $x = \pm a$ because the function $f(x)$ is defined to have a value “half way up the step” at these points. Let us examine the behavior of $f_\lambda(x)$ as $|\lambda| \to \infty$ in a region just one side of one of the discontinuities, that is, for $x \in (0, a)$. For a fixed λ, the difference, $f_\lambda(x) - f(x)$, oscillates above and below the value 0 as $x \to a$, attaining a maximum positive value at some point, say $x = x_\lambda$. Then the quantity $f_\lambda(x_\lambda) - f(x_\lambda)$ is called the *overshoot*.

As $|\lambda| \to \infty$, so the period of the oscillations tends to zero and so also $x_\lambda \to a$; however, the value of the overshoot $f_\lambda(x_\lambda) - f(x_\lambda)$ does not tend to zero but instead tends to a finite limit. The existence of this non-zero, finite, limiting value for the overshoot is known as the *Gibbs phenomenon.* This phenomenon also occurs in an almost identical manner in the Fourier synthesis of periodic functions using Fourier series.

2.9 Heisenberg's Uncertainty Principle

If $f \in L^2(\mathbb{R})$, then f and $F(k) = \mathscr{F}\{f(x)\}$ cannot both be essentially localized. In other words, it is not possible that the widths of the graphs of $|f(x)|^2$ and $|F(k)|^2$ can both be made arbitrarily small. This fact underlines the Heisenberg uncertainty principle in quantum mechanics and the bandwidth theorem in signal analysis. If $|f(x)|^2$ and $|F(k)|^2$ are interpreted as weighting functions, then the weighted means (averages) $<x>$ and $<k>$ of x and k are given by

$$<x> = \frac{1}{||f||_2^2} \int_{-\infty}^{\infty} x\,|f(x)|^2\,dx, \tag{2.9.1}$$

$$<k> = \frac{1}{||F||_2^2} \int_{-\infty}^{\infty} k\,|F(k)|^2\,dk. \tag{2.9.2}$$

Corresponding measures of the widths of these weight functions are given by the second moments about the respective means. Usually, it is convenient to define widths $\triangle x$ and $\triangle k$ by

$$(\triangle x)^2 = \frac{1}{||f||_2^2} \int_{-\infty}^{\infty} (x- <x>)^2\,|f(x)|^2\,dx, \tag{2.9.3}$$

$$(\triangle k)^2 = \frac{1}{||F||_2^2} \int_{-\infty}^{\infty} (k- <k>)^2\,|F(k)|^2\,dk. \tag{2.9.4}$$

The essence of the *Heisenberg principle* and the bandwidth theorems lies in the fact that the product $(\triangle x)(\triangle k)$ will never less than $\frac{1}{2}$. Indeed,

$$(\triangle x)(\triangle k) \geq \frac{1}{2}, \tag{2.9.5}$$

where equality in (2.9.5) holds only if $f(x)$ is a Gaussian function given by $f(x) = C\exp(-ax^2)$, $a > 0$.

We next state the Heisenberg inequality theorem as follows:

THEOREM 2.9.1** (Heisenberg Inequality).
If $f(x)$, $x\,f(x)$ and $k\,F(k)$ belong to $L^2(\mathbb{R})$ and $\sqrt{x}|f(x)\;\;| \to 0$ as $|x| \to \infty$, then

$$(\triangle x)^2(\triangle k)^2 \geq \frac{1}{4}, \tag{2.9.6}$$

where $(\triangle x)^2$ and $(\triangle k)^2$ are defined by (2.9.3) and (2.9.4), respectively. Equality in (2.9.6) holds only if $f(x)$ is a *Gaussian function* given by $f(x)=C\,e^{-ax^2}$, $a > 0$.

PROOF If the averages are $<x>$ and $<k>$, then the average location of $\exp(-i<k>x)f(x+<x>)$ is zero. Hence, it is sufficient to prove the theorem around the zero mean values, that is, $<x> = <k> =0$. Since $||f||_2 = ||F||_2$, we have

$$||f||_2^4(\triangle x)^2(\triangle k)^2 = \int_{-\infty}^{\infty} |xf(x)|^2 dx \int_{-\infty}^{\infty} |kF(k)|^2 dk.$$

Using $ikF(k) = \mathscr{F}\{f'(x)\}$ and the Parseval formula $||f'(x)||_2 = ||ikF(k)||_2$, we obtain

$$\begin{aligned}
||f||_2^4(\triangle x)^2(\triangle k)^2 &= \int_{-\infty}^{\infty} |xf(x)|^2 dx \int_{-\infty}^{\infty} |f'(x)|^2 dx \\
&\geq \left| \int_{-\infty}^{\infty} \left\{ xf(x)\, \overline{f'(x)} \right\} dx \right|^2 \ , \ \ \text{(see Debnath (2002))} \\
&\geq \left| \int_{-\infty}^{\infty} x.\frac{1}{2} \left\{ f'(x)\, \overline{f(x)} + \overline{f'(x)}\, f(x) \right\} \right|^2 \\
&= \frac{1}{4} \left[\int_{-\infty}^{\infty} x \left(\frac{d}{dx}|f|^2 \right) dx \right]^2 \\
&= \frac{1}{4} \left\{ \left[x|f(x)|^2 \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} |f|^2 dx \right\}^2 = \frac{1}{4}||f||_2^4 ,
\end{aligned}$$

in which $\sqrt{x}f(x) \to 0$ as $|x| \to \infty$ was used to eliminate the integrated term. This completes the proof.

If we assume $f'(x)$ is proportional to $x\, f(x)$, that is, $f'(x) = b\, x\, f(x)$, where b is a constant of proportionality, this leads to the *Gaussian signals*

$$f(x) = C \exp(-ax^2),$$

where C is a constant of integration and $a = -\frac{b}{2} > 0$. $\blacksquare$

In 1924, Heisenberg first formulated the uncertainty principle between the position and momentum in quantum mechanics. This principle has an important interpretation as an uncertainty of both the position and momentum of a particle described by a wave function $\psi \in L^2(\mathbb{R})$. In other words, it is not possible to determine the position and momentum of a particle exactly and simultaneously.

In signal processing, time and frequency concentrations of energy of a signal f are also governed by the Heisenberg uncertainty principle. The average or expectation values of time t and frequency ω, are respectively defined by

$$<t>= \frac{1}{||f||_2^2} \int_{-\infty}^{\infty} t|f(t)|^2 dt, \qquad <\omega>= \frac{1}{||F||_2^2} \int_{-\infty}^{\infty} \omega |F(\omega)|^2 d\omega, \quad (2.9.7)$$

where the energy of a signal $f(t)$ is well localized in time, and its Fourier transform $F(\omega)$ has an energy concentrated in a small frequency domain. The variances around these average values are given, respectively, by

$$\begin{aligned}
\sigma_t^2 &= \frac{1}{||f||_2^2} \int_{-\infty}^{\infty} (t- <t>)^2 |f(t)|^2 dt, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (2.9.8) \\
\sigma_\omega^2 &= \frac{1}{2\pi ||F||_2^2} \int_{-\infty}^{\infty} (\omega- <\omega>)^2 |F(\omega)|^2 d\omega.
\end{aligned}$$

Remarks:

1. In a time-frequency analysis of signals, the measure of the resolution of a signal f in the time or frequency domain is given by σ_t and σ_ω. Then, the joint resolution is given by the product (σ_t) (σ_ω) which is governed by the Heisenberg uncertainty principle. In other words, the product (σ_t) (σ_ω) cannot be arbitrarily small and is always greater than the minimum value $\frac{1}{2}$ which is attained for the Gaussian signal.

2. In many applications in science and engineering, signals with a high concentration of energy in the time and frequency domains are of special interest. The uncertainty principle can also be interpreted as a measure of this concentration of the second moment of $f^2(t)$ and its energy spectrum $F^2(\omega)$.

2.10 Applications of Fourier Transforms to Ordinary Differential Equations

We consider the nth-order linear ordinary differential equation with constant coefficients

$$Ly(x) = f(x), \tag{2.10.1}$$

where L is the nth-order differential operator given by

$$L \equiv a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0, \tag{2.10.2}$$

where $a_n, a_{n-1}, \ldots, a_1, a_0$ are constants, $D \equiv \frac{d}{dx}$ and $f(x)$ is a given function.

Application of the Fourier transform to both sides of (2.10.1) gives

$$[a_n(ik)^n + a_{n-1}(ik)^{n-1} + \cdots + a_1(ik) + a_0]Y(k) = F(k),$$

where $\mathscr{F}\{y(x)\} = Y(k)$ and $\mathscr{F}\{f(x)\} = F(k)$.

Or, equivalently

$$P(ik)Y(k) = F(k),$$

where

$$P(z) = \sum_{r=0}^{n} a_r z^r.$$

Thus,

$$Y(k) = \frac{F(k)}{P(ik)} = F(k)Q(k), \tag{2.10.3}$$

where $Q(k) = \frac{1}{P(ik)}$.

Applying the Convolution Theorem 2.5.5 to (2.10.3) gives the formal solution

$$y(x) = \mathscr{F}^{-1}\{F(k)\,Q(k)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\xi)q(x-\xi)d\xi, \tag{2.10.4}$$

provided $q(x) = \mathscr{F}^{-1}\{Q(k)\}$ is known explicitly.

In order to give a physical interpretation of the solution (2.10.4), we consider the differential equation with a suddenly applied impulse function $f(x) = \delta(x)$ so that

$$L\{G(x)\} = \delta(x). \tag{2.10.5}$$

The solution of this equation can be written from the inversion of (2.10.3) in the form

$$G(x) = \mathscr{F}^{-1}\left\{\frac{1}{\sqrt{2\pi}}\,Q(k)\right\} = \frac{1}{\sqrt{2\pi}}\,q(x). \tag{2.10.6}$$

Thus, the solution (2.10.4) takes the form

$$y(x) = \int_{-\infty}^{\infty} f(\xi)G(x-\xi)d\xi. \tag{2.10.7}$$

Clearly, $G(x)$ behaves like a *Green's function*, that is, it is the response to a *unit impulse*. In any physical system, $f(x)$ usually represents the *input function*, while $y(x)$ is referred to as the *output* obtained by the superposition principle. The Fourier transform of $\{\sqrt{2\pi}G(x)\} = q(x)$ is called the *admittance*. In order to find the response to a given input, we determine the Fourier transform of the input function, multiply the result by the admittance, and then apply the inverse Fourier transform to the product so obtained.

We illustrate these ideas by solving a simple problem in the electrical circuit theory.

Example 2.10.1 (*Electric Current in a Simple Circuit*).
The current $I(t)$ in a simple circuit containing the resistance R and inductance L satisfies the equation

$$L\,\frac{dI}{dt} + RI = E(t), \tag{2.10.8}$$

where $E(t)$ is the applied electromagnetic force and R and L are constants.

With $E(t) = E_0\exp(-a|t|)$, we use the Fourier transform with respect to time t to obtain

$$(ikL + R)\hat{I}(k) = E_0\sqrt{\frac{2}{\pi}}\frac{a}{(a^2+k^2)}.$$

Or,

$$\hat{I}(k) = \frac{aE_0}{iL}\sqrt{\frac{2}{\pi}}\frac{1}{\left(k - \frac{Ri}{L}\right)\,(k^2+a^2)},$$

where $\mathscr{F}\{I(t)\} = \hat{I}(k)$. The inverse Fourier transform gives

$$I(t) = \frac{aE_0}{i\pi L}\int_{-\infty}^{\infty} \frac{\exp(ikt)dk}{\left(k - \frac{Ri}{L}\right)(k^2 + a^2)}. \tag{2.10.9}$$

This integral can be evaluated by the Cauchy Residue Theorem. For $t > 0$

$$\begin{aligned} I(t) &= \frac{aE_0}{i\pi L}\cdot 2\pi i\left[\text{Residue at}\, k = \frac{Ri}{L} + \text{Residue at}\, k = ia\right] \\ &= \frac{2aE_0}{L}\left[\frac{e^{-\frac{R}{L}t}}{\left(a^2 - \frac{R^2}{L^2}\right)} - \frac{e^{-at}}{2a\left(a - \frac{R}{L}\right)}\right] \\ &= E_0\left[\frac{e^{-at}}{R - aL} - \frac{2aLe^{-\frac{R}{L}t}}{R^2 - a^2L^2}\right]. \end{aligned} \tag{2.10.10}$$

Similarly, for $t < 0$, the Residue Theorem gives

$$\begin{aligned} I(t) &= -\frac{aE_0}{i\pi L}\cdot 2\pi i[\text{Residue at}\, k = -ia] \\ &= -\frac{2aE_0}{L}\left[\frac{-Le^{at}}{(aL + R)2a}\right] = \frac{E_0 e^{at}}{(aL + R)}. \end{aligned} \tag{2.10.11}$$

At $t = 0$, the current is continuous and therefore,

$$I(0) = \lim_{t\to 0} I(t) = \frac{E_0}{R + aL}.$$

If $E(t) = \delta(t)$, then $\hat{E}(k) = \frac{1}{\sqrt{2\pi}}$ and the solution is obtained by using the inverse Fourier transform

$$I(t) = \frac{1}{2\pi i L}\int_{-\infty}^{\infty} \frac{e^{ikt}}{k - \frac{iR}{L}}\, dk,$$

which is, by the Theorem of Residues,

$$\begin{aligned} &= \frac{1}{L}[\text{Residue at}\, k = iR/L] \\ &= \frac{1}{L}\exp\left(-\frac{Rt}{L}\right). \end{aligned} \tag{2.10.12}$$

Thus, the current tends to zero as $t \to \infty$ as expected. ▯

Example 2.10.2 Find the solution of the ordinary differential equation

$$-\frac{d^2u}{dx^2} + a^2u = f(x), \quad -\infty < x < \infty \tag{2.10.13}$$

by the Fourier transform method.

Application of the Fourier transform to (2.10.13) gives

$$U(k) = \frac{F(k)}{k^2 + a^2}.$$

This can readily be inverted by the Convolution Theorem 2.5.5 to obtain

$$u(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi) g(x-\xi) d\xi, \tag{2.10.14}$$

where $g(x) = \mathscr{F}^{-1}\left\{\frac{1}{k^2+a^2}\right\} = \frac{1}{a}\sqrt{\frac{\pi}{2}}\exp(-a|x|)$ by Example 2.3.2. Thus, the final solution is

$$u(x) = \frac{1}{2a} \int_{-\infty}^{\infty} f(\xi) e^{-a|x-\xi|}\, d\xi. \tag{2.10.15}$$

$\square$

Example 2.10.3 (*The Bernoulli-Euler Beam Equation*).
We consider the vertical deflection $u(x)$ of an infinite beam on an elastic foundation under the action of a prescribed vertical load $W(x)$. The deflection $u(x)$ satisfies the ordinary differential equation

$$EI\frac{d^4u}{dx^4} + \kappa u = W(x), \quad -\infty < x < \infty. \tag{2.10.16}$$

where EI is the flexural rigidity and κ is the foundation modulus of the beam. We find the solution assuming that $W(x)$ has a compact support and u, u', u'', u''' all tend to zero as $|x| \to \infty$.

We first rewrite (2.10.16) as

$$\frac{d^4u}{dx^4} + a^4 u = w(x) \tag{2.10.17}$$

where $a^4 = \kappa/EI$ and $w(x) = W(x)/EI$. Use of the Fourier transform to (2.10.17) gives

$$U(k) = \frac{W(k)}{k^4 + a^4}.$$

The inverse Fourier transform gives the solution

$$\begin{aligned} u(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{W(k)}{k^4 + a^4} e^{ikx}\, dk \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ikx}}{k^4 + a^4} dk \int_{-\infty}^{\infty} w(\xi) e^{-ik\xi}\, d\xi \\ &= \int_{-\infty}^{\infty} w(\xi) G(\xi, x)\, d\xi, \end{aligned} \tag{2.10.18}$$

where

$$G(\xi, x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ik(x-\xi)}}{k^4 + a^4} dk = \frac{1}{\pi} \int_{0}^{\infty} \frac{\cos k(x-\xi)\, dk}{k^4 + a^4}. \tag{2.10.19}$$

The integral can be evaluated by the Theorem of Residues or by using the table of Fourier integrals. We simply state the result

$$G(\xi, x) = \frac{1}{2a^3} \exp\left(-\frac{a}{\sqrt{2}}|x-\xi|\right) \sin\left[\frac{a(x-\xi)}{\sqrt{2}} + \frac{\pi}{4}\right]. \tag{2.10.20}$$

In particular, we find the explicit solution due to a concentrated load of unit strength acting at some point x_0, that is, $w(x) = \delta(x - x_0)$. Then the solution for this case becomes

$$u(x) = \int_{-\infty}^{\infty} \delta(\xi - x_0) G(x, \xi)\, d\xi = G(x, x_0). \tag{2.10.21}$$

Thus, the kernel $G(x, \xi)$ involved in the solution (2.10.18) has the physical significance of being the deflection, as a function of x, due to a unit point load acting at ξ. Thus, the deflection due to a point load of strength $w(\xi)\, d\xi$ at ξ is $w(\xi)\, d\xi \cdot G(x, \xi)$, and hence, (2.10.18) represents the superposition of all such incremental deflections.

The reader is referred to a more general dynamic problem of an infinite Bernoulli-Euler beam with damping and elastic foundation that has been solved by Stadler and Shreeves (1970), and also by Sheehan and Debnath (1972). These authors used the Fourier-Laplace transform method to determine the steady state and the transient solutions of the beam problem. ▯

2.11 Solutions of Integral Equations

The method of Fourier transforms can be used to solve simple integral equations of the convolution type. We illustrate the method by examples.

We first solve the *Fredholm integral equation* with convolution kernel in the form

$$\int_{-\infty}^{\infty} f(t)g(x-t)\,dt + \lambda f(x) = u(x), \tag{2.11.1}$$

where $g(x)$ and $u(x)$ are given functions and λ is a known parameter.

Application of the Fourier transform to (2.11.1) gives

$$\sqrt{2\pi}F(k)G(k) + \lambda F(k) = U(k).$$

Or,

$$F(k) = \frac{U(k)}{\sqrt{2\pi}G(k)+\lambda}. \tag{2.11.2}$$

The inverse Fourier transform leads to a formal solution

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{U(k)e^{ikx}dk}{\sqrt{2\pi}G(k)+\lambda}. \tag{2.11.3}$$

In particular, if $g(x) = \frac{1}{x}$ so that

$$G(k) = -i\sqrt{\frac{\pi}{2}}\,\mathrm{sgn}\,\mathrm{k},$$

then the solution becomes

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{U(k)e^{ikx}dk}{\lambda - i\pi\,\mathrm{sgn}\,\mathrm{k}}. \tag{2.11.4}$$

If $\lambda = 1$ and $g(x) = \frac{1}{2}\left(\frac{x}{|x|}\right)$ so that $G(k) = \frac{1}{\sqrt{2\pi}}\frac{1}{(ik)}$, solution (2.11.3) reduces to the form

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} (ik)\frac{U(k)e^{ikx}\,dk}{(1+ik)} \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathscr{F}\{u'(x)\}\mathscr{F}\{\sqrt{2\pi}\,e^{-x}\}e^{ikx}\,dk \\ &= u'(x) * \sqrt{2\pi}\,e^{-x} = \int_{-\infty}^{\infty} u'(\xi)\exp(\xi - x)\,d\xi. \end{aligned} \tag{2.11.5}$$

Example 2.11.1 Find the solution of the integral equation

$$\int_{-\infty}^{\infty} f(x-\xi)f(\xi)\,d\xi = \frac{1}{x^2+a^2}. \tag{2.11.6}$$

Application of the Fourier transform gives

$$\sqrt{2\pi}F(k)\,F(k) = \sqrt{\frac{\pi}{2}}\,\frac{e^{-a|k|}}{a}.$$

Or,

$$F(k) = \frac{1}{\sqrt{2a}}\exp\left\{-\frac{1}{2}a|k|\right\}. \tag{2.11.7}$$

The inverse Fourier transform gives the solution

$$\begin{aligned}
f(x) &= \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2a}}\int_{-\infty}^{\infty}\exp\left(ikx-\frac{1}{2}a|k|\right)\,dk \\
&= \frac{1}{2\sqrt{\pi a}}\left[\int_0^{\infty}\exp\left\{-k\left(\frac{a}{2}+ix\right)\right\}\,dk + \int_0^{\infty}\exp\left\{-k\left(\frac{a}{2}-ix\right)\right\}\,dk\right] \\
&= \frac{1}{2\sqrt{\pi a}}\left[\frac{4a}{(4x^2+a^2)}\right] = \sqrt{\frac{a}{\pi}}\cdot\frac{2}{(4x^2+a^2)}.
\end{aligned}$$

Using the table B-1 of Fourier transform (see No. 4), we also get the same result :

$$f(x) = \mathscr{F}^{-1}\{F(k)\} = \sqrt{\frac{a}{x}}\frac{2}{4x^2+a^2}.$$

$\square$

Example 2.11.2 Solve the integral equation

$$\int_{-\infty}^{\infty}\frac{f(t)\,dt}{(x-t)^2+a^2} = \frac{1}{(x^2+b^2)},\ b>a>0. \tag{2.11.8}$$

Taking the Fourier transform, we obtain

$$\sqrt{2\pi}\,F(k)\,\mathscr{F}\left\{\frac{1}{x^2+a^2}\right\} = \sqrt{\frac{\pi}{2}}\,\frac{e^{-b|k|}}{b},$$

or,

$$\sqrt{2\pi}\,F(k)\sqrt{\frac{\pi}{2}}\cdot\frac{e^{-a|k|}}{a} = \sqrt{\frac{\pi}{2}}\,\frac{e^{-b|k|}}{b}.$$

Thus,

$$F(k) = \frac{1}{\sqrt{2\pi}} \left(\frac{a}{b}\right) \exp\{-|k|(b-a)\}. \tag{2.11.9}$$

The inverse Fourier transform leads to the solution

$$\begin{aligned}
f(x) &= \frac{a}{2\pi b} \int_{-\infty}^{\infty} \exp[ikx - |k|(b-a)]dk \\
&= \frac{a}{2\pi b} \left[\int_{0}^{\infty} \exp[-k\{(b-a)+ix\}]dk + \int_{0}^{\infty} \exp[-k\{(b-a)-ix\}]\right] dk \\
&= \frac{a}{2\pi b} \left[\frac{1}{(b-a)+ix} + \frac{1}{(b-a)-ix}\right] \\
&= \left(\frac{a}{\pi b}\right) \frac{(b-a)}{(b-a)^2 + x^2}. \qquad (2.11.10)
\end{aligned}$$

□

Example 2.11.3 Solve the integral equation

$$f(x) + 4 \int_{-\infty}^{\infty} e^{-a|x-t|} f(t)dt = g(x). \tag{2.11.11}$$

Application of the Fourier transform gives

$$\begin{aligned}
F(k) + 4\sqrt{2\pi}F(k) \cdot \frac{2a}{\sqrt{2\pi}(a^2+k^2)} &= G(k) \\
F(k) &= \frac{(a^2+k^2)}{a^2+k^2+8a}\, G(k). \qquad (2.11.12)
\end{aligned}$$

The inverse Fourier transform gives

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{(a^2+k^2)G(k)}{a^2+k^2+8a} e^{ikx} dk. \tag{2.11.13}$$

In particular, if $a=1$ and $g(x) = e^{-|x|}$ so that $G(k) = \sqrt{\frac{2}{\pi}} \frac{1}{1+k^2}$, then solution (2.11.13) becomes

$$f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{ikx}}{k^2+3^2}\, dk. \tag{2.11.14}$$

For $x > 0$, we use a semicircular closed contour in the lower half of the complex plane to evaluate (2.11.14). It turns out that

$$f(x) = \frac{1}{3}\, e^{-3x}. \tag{2.11.15}$$

Similarly, for $x<0$, a semicircular closed contour in the upper half of the complex plane is used to evaluate (2.11.14) so that

$$f(x)=\frac{1}{3}\,e^{3x},\quad x<0. \tag{2.11.16}$$

Thus, the final solution is

$$f(x)=\frac{1}{3}\exp(-3|x|). \tag{2.11.17}$$

□

2.12 Solutions of Partial Differential Equations

In this section we illustrate how the Fourier transform method can be used to obtain the solution of boundary value and initial value problems for linear partial differential equations of different kinds.

Example 2.12.1 (*Dirichlet's Problem in the Half-Plane*).
We consider the solution of the Laplace equation in the half-plane

$$u_{xx}+u_{yy}=0,\quad -\infty<x<\infty,\quad y\geq 0, \tag{2.12.1}$$

with the boundary conditions

$$u(x,0)=f(x),\qquad -\infty<x<\infty, \tag{2.12.2}$$
$$u(x,y)\to 0 \text{ as } |x|\to\infty,\quad y\to\infty. \tag{2.12.3}$$

We introduce the Fourier transform with respect to x

$$U(k,y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}u(x,y)dx \tag{2.12.4}$$

so that (2.12.1)–(2.12.3) becomes

$$\frac{d^2U}{dy^2}-k^2U=0, \tag{2.12.5}$$

$$U(k,0)=F(k),\quad U(k,y)\to 0\qquad \text{as } y\to\infty. \tag{2.12.6ab}$$

Thus, the solution of this transformed system is

$$U(k,y)=F(k)e^{-|k|y}. \tag{2.12.7}$$

Application of the Convolution Theorem 2.5.5 gives the solution

$$u(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi) g(x-\xi) d\xi, \tag{2.12.8}$$

where

$$g(x) = \mathscr{F}^{-1}\{e^{-|k|y}\} = \sqrt{\frac{2}{\pi}} \frac{y}{(x^2+y^2)}. \tag{2.12.9}$$

Consequently, the solution (2.12.8) becomes

$$u(x,y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi)d\xi}{(x-\xi)^2+y^2}, \quad y > 0. \tag{2.12.10}$$

This is the well-known *Poisson integral formula* in the half-plane. It is noted that

$$\lim_{y\to 0^+} u(x,y) = \int_{-\infty}^{\infty} f(\xi) \left[\lim_{y\to 0^+} \frac{y}{\pi} \cdot \frac{1}{(x-\xi)^2+y^2} \right] d\xi = \int_{-\infty}^{\infty} f(\xi)\delta(x-\xi)d\xi, \tag{2.12.11}$$

where Cauchy's definition of the delta function is used, that is,

$$\delta(x-\xi) = \lim_{y\to 0^+} \frac{y}{\pi} \cdot \frac{1}{(x-\xi)^2+y^2}. \tag{2.12.12}$$

This may be recognized as a solution of the Laplace equation for a dipole source at $(x,y) = (\xi, 0)$.

In particular, when

$$f(x) = T_0 H(a - |x|), \tag{2.12.13}$$

the solution (2.12.10) reduces to

$$\begin{aligned} u(x,y) &= \frac{yT_0}{\pi} \int_{-a}^{a} \frac{d\xi}{(\xi-x)^2+y^2} \\ &= \frac{T_0}{\pi} \left[\tan^{-1}\left(\frac{x+a}{y}\right) - \tan^{-1}\left(\frac{x-a}{y}\right) \right] \\ &= \frac{T_0}{\pi} \tan^{-1}\left(\frac{2ay}{x^2+y^2-a^2}\right). \end{aligned} \tag{2.12.14}$$

The curves in the upper half-plane for which the steady state temperature is constant are known as *isothermal curves.* In this case, these curves represent a family of circular arcs

$$x^2 + y^2 - \alpha y = a^2 \tag{2.12.15}$$

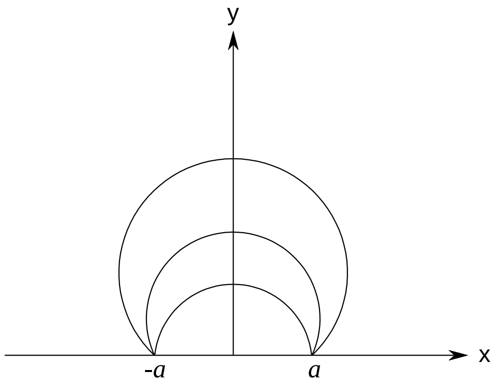

Figure 2.9 A family of circular arcs.

with centers on the y-axis and the fixed end points on the x-axis at $x=\pm a$. The graphs of the arcs are is displayed in Figure 2.9.

Another special case deals with

$$f(x)=\delta(x). \tag{2.12.16}$$

The solution for this case follows from (2.12.10) and is

$$u(x,y)=\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{\delta(\xi)d\xi}{(x-\xi)^2+y^2}=\frac{y}{\pi}\frac{1}{(x^2+y^2)}. \tag{2.12.17}$$

Further, we can readily deduce the solution of the *Neumann problem* in the half-plane from the solution of the Dirichlet problem. ▯

Example 2.12.2 (*Neumann's Problem in the Half-Plane*).
Find a solution of the Laplace equation

$$u_{xx}+u_{yy}=0, \quad -\infty<x<\infty, \quad y>0, \tag{2.12.18}$$

with the boundary condition

$$u_y(x,0)=f(x), \quad -\infty<x<\infty. \tag{2.12.19}$$

This condition specifies the normal derivative on the boundary, and physically, it describes the fluid flow or, heat flux at the boundary.

We define a new function $v(x,y)=u_y(x,y)$ so that

$$u(x,y)=\int^{y} v(x,\eta)d\eta, \tag{2.12.20}$$

where an arbitrary constant can be added to the right-hand side. Clearly, the function v satisfies the Laplace equation

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=\frac{\partial^2 u_y}{\partial x^2}+\frac{\partial^2 u_y}{\partial y^2}=\frac{\partial}{\partial y}(u_{xx}+u_{yy})=0,$$

with the boundary condition

$$v(x,0)=u_y(x,0)=f(x) \text{ for } -\infty<x<\infty.$$

Thus, $v(x,y)$ satisfies the Laplace equation with the Dirichlet condition on the boundary. Obviously, the solution is given by (2.12.10); that is,

$$v(x,y)=\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(\xi)d\xi}{(x-\xi)^2+y^2}. \tag{2.12.21}$$

Then the solution $u(x,y)$ can be obtained from (2.12.20) in the form

$$\begin{aligned}u(x,y) &= \int^{y} v(x,\eta)d\eta=\frac{1}{\pi}\int^{y}\eta\,d\eta\int_{-\infty}^{\infty}\frac{f(\xi)d\xi}{(x-\xi)^2+\eta^2}\\ &=\frac{1}{\pi}\int_{-\infty}^{\infty}f(\xi)d\xi\int^{y}\frac{\eta\,d\eta}{(x-\xi)^2+\eta^2}, \quad y>0\\ &=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(\xi)\log[(x-\xi)^2+y^2]d\xi, \qquad (2.12.22)\end{aligned}$$

where an arbitrary constant can be added to this solution. In other words, the solution of any Neumann problem is uniquely determined up to an arbitrary constant. $\square$

Example 2.12.3 (*The Cauchy Problem for the Diffusion Equation*).
We consider the initial value problem for a one-dimensional diffusion equation with no sources or sinks

$$u_t=\kappa u_{xx}, \quad -\infty<x<\infty, \quad t>0, \tag{2.12.23}$$

where κ is a diffusivity constant with the initial condition

$$u(x,0)=f(x), \quad -\infty<x<\infty. \tag{2.12.24}$$

We solve this problem using the Fourier transform in the space variable x defined by (2.12.4). Application of this transform to (2.12.23)–(2.12.24) gives

$$U_t = -\kappa k^2 U, \quad t>0, \tag{2.12.25}$$
$$U(k,0) = F(k). \tag{2.12.26}$$

The solution of the transformed system is

$$U(k,t) = F(k)\, e^{-\kappa k^2 t}. \tag{2.12.27}$$

The inverse Fourier transform gives the solution

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) \exp[(ikx - \kappa k^2 t)] dk$$

which is, by the Convolution Theorem 2.5.5,

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi) g(x-\xi) d\xi, \tag{2.12.28}$$

where

$$g(x) = \mathscr{F}^{-1}\{e^{-\kappa k^2 t}\} = \frac{1}{\sqrt{2\kappa t}} \exp\left(-\frac{x^2}{4\kappa t}\right), \text{ by (2.3.5).}$$

Thus, solution (2.12.28) becomes

$$u(x,t) = \frac{1}{\sqrt{4\pi\kappa t}} \int_{-\infty}^{\infty} f(\xi) \exp\left[-\frac{(x-\xi)^2}{4\kappa t}\right] d\xi. \tag{2.12.29}$$

The integrand involved in the solution consists of the initial value $f(x)$ and *Green's function* (or, *elementary solution*) $G(x-\xi,t)$ of the diffusion equation for the infinite interval:

$$G(x-\xi,t) = \frac{1}{\sqrt{4\pi\kappa t}} \exp\left[-\frac{(x-\xi)^2}{4\kappa t}\right]. \tag{2.12.30}$$

So, in terms of $G(x-\xi,t)$, solution (2.12.29) can be written as

$$u(x,t) = \int_{-\infty}^{\infty} f(\xi) G(x-\xi,t) d\xi \tag{2.12.31}$$

so that, in the limit as $t \to 0+$, this formally becomes

$$u(x,0) = f(x) = \int_{-\infty}^{\infty} f(\xi) \lim_{t\to 0+} G(x-\xi,t) d\xi.$$

The limit of $G(x-\xi,t)$ represents the Dirac delta function

$$\delta(x-\xi) = \lim_{t\to 0+} \frac{1}{2\sqrt{\pi\kappa t}} \exp\left[-\frac{(x-\xi)^2}{4\,\kappa t}\right]. \tag{2.12.32}$$

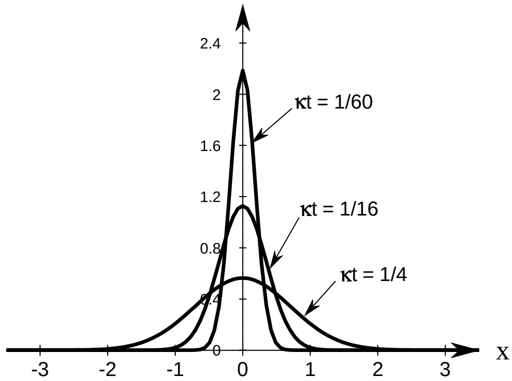

Figure 2.10 Graphs of $G(x,t)$ against x.

Graphs of $G(x,t)$ are shown in Figure 2.10 for different values of κt.

It is important to point out that the integrand in (2.12.31) consists of the initial temperature distribution $f(x)$ and Green's function $G(x-\xi,t)$ which represents the temperature response along the rod at time t due to an initial unit impulse of heat at $x=\xi$. The physical meaning of the solution (2.12.31) is that the initial temperature distribution $f(x)$ is decomposed into a spectrum of impulses of magnitude $f(\xi)$ at each point $x=\xi$ to form the resulting temperature $f(\xi)G(x-\xi,t)$. Thus, the resulting temperature is integrated to find solution (2.12.31). This is called the *principle of integral superposition.*

We make the change of variable

$$\frac{\xi-x}{2\sqrt{\kappa t}}=\zeta, \quad d\zeta=\frac{d\xi}{2\sqrt{\kappa t}}$$

to express solution (2.12.29) in the form

$$u(x,t)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(x+2\sqrt{\kappa t}\,\zeta)\exp(-\zeta^2)d\zeta. \tag{2.12.33}$$

The integral solution (2.12.33) or (2.12.29) is called the *Poisson integral representation* of the temperature distribution. This integral is convergent for all time $t>0$, and the integrals obtained from (2.12.33) by differentiation under the integral sign with respect to x and t are uniformly convergent in the neighborhood of the point (x,t). Hence, the solution $u(x,t)$ and its derivatives of all orders exist for $t>0$.

Finally, we consider a special case involving discontinuous initial condition in the form

$$f(x)=T_0H(x)\,, \tag{2.12.34}$$

where T_0 is a constant. In this case, solution (2.12.29) becomes

$$u(x,t) = \frac{T_0}{2\sqrt{\pi\kappa t}} \int_0^\infty \exp\left[-\frac{(x-\xi)^2}{4\,\kappa t}\right] d\xi. \tag{2.12.35}$$

Introducing the change of variable $\eta = \frac{\xi - x}{2\sqrt{\kappa t}}$, we can express solution (2.12.35) in the form

$$\begin{aligned} u(x,t) &= \frac{T_0}{\sqrt{\pi}} \int_{-x/2\sqrt{\kappa t}}^{\infty} e^{-\eta^2}\, d\eta = \frac{T_0}{2}\, erfc\left(-\frac{x}{2\sqrt{\kappa t}}\right) \\ &= \frac{T_0}{2}\left[1 + erf\left(\frac{x}{2\sqrt{\kappa t}}\right)\right]. \end{aligned} \tag{2.12.36}$$

The solution given by equation (2.12.36) with $T_0 = 1$ is shown in Figure 2.11.

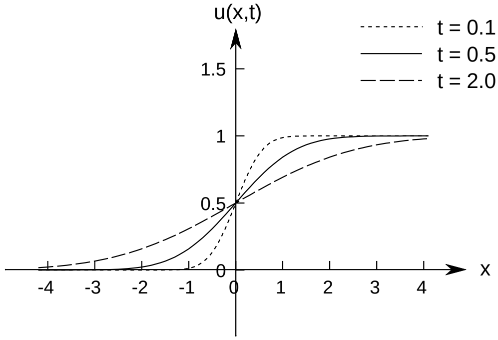

Figure 2.11 The time development of solution (2.12.36).

▯

If $f(x) = \delta(x)$, then the fundamental solution (2.12.29) is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi\,\kappa\,t}} \exp\left(-\frac{x^2}{4\kappa\,t}\right).$$

Example 2.12.4 (*The Cauchy Problem for the Wave Equation*).
Obtain the d'Alembert solution of the initial value problem for the wave equation

$$u_{tt} = c^2 u_{xx}, \quad -\infty < x < \infty, \quad t > 0, \tag{2.12.37}$$

with the arbitrary but fixed initial data

$$u(x,0)=f(x), \quad u_t(x,0)=g(x), \quad -\infty<x<\infty. \tag{2.12.38ab}$$

Application of the Fourier transform $\mathscr{F}\{u(x,t)\}=U(k,t)$ to this system gives

$$\frac{d^2U}{dt^2}+c^2k^2U=0,$$
$$U(k,0)=F(k), \quad \left(\frac{dU}{dt}\right)_{t=0}=G(k).$$

The solution of the transformed system is

$$U(k,t)=A\ e^{ickt}+B\ e^{-ickt},$$

where A and B are constants to be determined from the transformed data so that $A+B=F(k)$ and $A-B=\frac{1}{ikc}G(k)$. Solving for A and B, we obtain

$$U(k,t)=\frac{1}{2}F(k)(e^{ickt}+e^{-ickt})+\frac{G(k)}{2ick}(e^{ickt}-e^{-ickt}). \tag{2.12.39}$$

Thus, the inverse Fourier transform of (2.12.39) yields the solution

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)\{e^{ik(x+ct)}+e^{ik(x-ct)}\}dk\right] \\ &\quad +\frac{1}{2c}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{G(k)}{ik}\{e^{ik(x+ct)}-e^{ik(x-ct)}\}dk\right]. \end{aligned} \tag{2.12.40}$$

We use the following results

$$f(x)=\mathscr{F}^{-1}\{F(k)\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}F(k)dk,$$

$$g(x)=\mathscr{F}^{-1}\{G(k)\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}G(k)dk,$$

to obtain the solution in the final form

$$\begin{aligned} u(x,t) &= \frac{1}{2}[f(x-ct)+f(x+ct)]+\frac{1}{2c}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} G(k)dk\int_{x-ct}^{x+ct} e^{ik\xi}d\xi \\ &= \frac{1}{2}[f(x-ct)+f(x+ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct} d\xi\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ik\xi}G(k)dk\right] \\ &= \frac{1}{2}[f(x-ct)+f(x+ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)d\xi. \end{aligned} \tag{2.12.41}$$

This is the well-known *d'Alembert solution* of the wave equation.

The method and the form of the solution reveal several important features of the wave equation. First, the method of solution essentially proves the existence of the d'Alembert solution and the solution is unique provided $f(x)$ is twice continuously differentiable and $g(x)$ is continuously differentiable. Second, the terms involving $f(x \pm ct)$ in (2.12.41) show that disturbances are propagated along the characteristics with constant velocity c. Both terms combined together suggest that the value of the solution at position x and at time t depends only on the initial values of $f(x)$ at $x - ct$ and $x + ct$ and the values of $g(x)$ between these two points. The interval $(x - ct, x + ct)$ is called the *domain of dependence* of the variable (x, t). Finally, the solution depends continuously on the initial data, that is, the problem is well posed. In other words, a small change in either $f(x)$ or $g(x)$ results in a correspondingly small change in the solution $u(x, t)$.

In particular, if $f(x) = \exp(-x^2)$ and $g(x) \equiv 0$, the time development of solution (2.12.41) with $c = 1$ is shown in Figure 2.12. In this case, the solution becomes

$$u(x,t) = \frac{1}{2}[e^{-(x-t)^2} + e^{-(x+t)^2}]. \tag{2.12.42}$$

As shown in Figure 2.12, the initial form $f(x) = \exp(-x^2)$ is found to split into two similar waves propagating in opposite direction with unit velocity.
□

Example 2.12.5 (*The Schrödinger Equation in Quantum Mechanics*).
The time-dependent Schrödinger equation of a particle of mass m is

$$i\hbar\,\psi_t = \left[V(x) - \frac{\hbar^2}{2m}\nabla^2\right]\psi = H\psi, \tag{2.12.43}$$

where $h = 2\pi\hbar$ is the *Planck constant*, $\psi(\mathbf{x}, t)$ is the wave function, $V(x)$ is the potential, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the three-dimensional *Laplacian*, and H is the *Hamiltonian.*

If $V(\mathbf{x}) = \text{constant} = V$, we can seek a *plane wave solution* of the form

$$\psi(\mathbf{x}, t) = A\exp[i(\boldsymbol{\kappa}\cdot\mathbf{x} - \omega t)], \tag{2.12.44}$$

where A is a constant amplitude, $\boldsymbol{\kappa} = (k, l, m)$ is the wavenumber vector, and ω is the frequency.

Substituting this solution into (2.12.43), we conclude that this solution is possible provided the following relation is satisfied:

$$i\hbar(-i\omega) = V - \frac{\hbar^2}{2m}(i\kappa)^2, \quad \kappa^2 = k^2 + l^2 + m^2.$$

Or,

$$\hbar\omega = V + \frac{\hbar^2\kappa^2}{2m}. \tag{2.12.45}$$

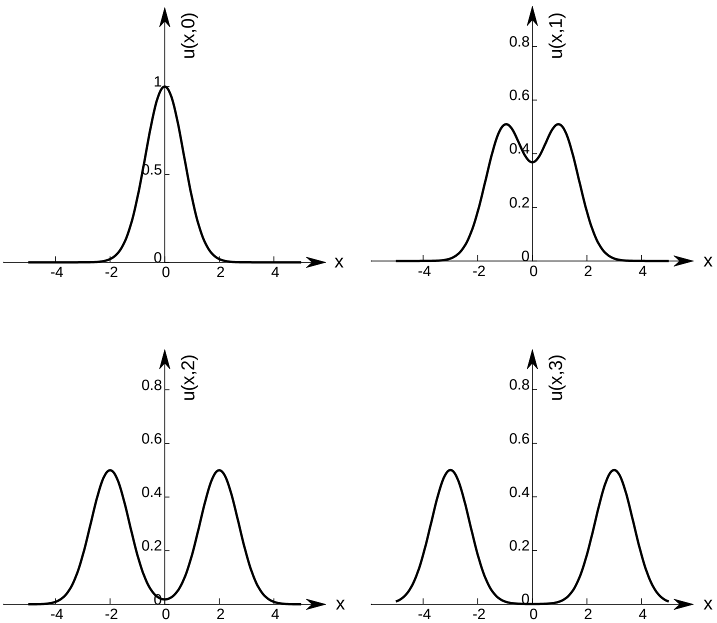

Figure 2.12 The time development of solution (2.12.42).

This is called the *dispersion relation* and shows that the sum of the potential energy V and the kinetic energy $\frac{(\hbar\kappa)^2}{2m}$ is equal to the total energy $\hbar\omega$. Further, the kinetic energy

$$K.E. = \frac{1}{2m}(\hbar\kappa)^2 = \frac{p^2}{2m}, \tag{2.12.46}$$

where $p = \hbar\kappa$ is the momentum of the particle.

The phase velocity, C_p and the group velocity, C_g of the wave are defined by

$$C_p = \frac{\omega}{\kappa}\,\hat{\boldsymbol{\kappa}}, \qquad C_g = \nabla_{\boldsymbol{\kappa}}\omega(\boldsymbol{\kappa}), \tag{2.12.47ab}$$

where $\boldsymbol{\kappa}$ is the wavenumber vector and $\kappa = |\boldsymbol{\kappa}|$ and $\hat{\boldsymbol{\kappa}}$ is the unit wavenumber vector.

In the one-dimensional case, the phase velocity is

$$C_p = \frac{\omega}{k} \tag{2.12.48}$$

and the group velocity is

$$C_g = \frac{\partial\omega}{\partial k} = \frac{\hbar k}{m} = \frac{p}{m} = \frac{mv}{v} = v. \tag{2.12.49}$$

This shows that the group velocity is equal to the classical particle velocity v.

We now use the Fourier transform method to solve the one-dimensional Schrödinger equation for a free particle ($V \equiv 0$), that is,

$$i\hbar\,\psi_t = -\frac{\hbar^2}{2m}\psi_{xx}, \quad -\infty < x < \infty, \quad t > 0, \tag{2.12.50}$$

$$\psi(x,0) = \psi_0(x), \quad -\infty < x < \infty, \tag{2.12.51}$$

$$\psi(x,t) \to 0 \quad \text{as } |x| \to \infty. \tag{2.12.52}$$

Application of the Fourier transform to (2.12.50)–(2.12.52) gives

$$\Psi_t = -\frac{i\hbar k^2}{2m}\Psi, \qquad \Psi(k,0) = \Psi_0(k). \tag{2.12.53}$$

The solution of this transformed system is

$$\Psi(k,t) = \Psi_0(k)\exp(-i\alpha k^2 t), \quad \alpha = \frac{\hbar}{2m}. \tag{2.12.54}$$

The inverse Fourier transform gives the formal solution

$$\psi(x,t) = \mathscr{F}^{-1}\{\Psi_0(k)\exp(-i\alpha k^2 t)\},$$

which is, by the convolution theorem 2.5.5,

$$\psi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Psi_0(\xi)g(x-\xi)d\xi, \tag{2.12.55}$$

where

$$g(x) = \mathscr{F}^{-1}\{\exp(-i\alpha k^2 t)\} = \frac{1}{\sqrt{2i\alpha t}}\exp(-\frac{x^2}{4i\alpha t}).$$

Consequently,

$$\psi(x,t) = \frac{1}{\sqrt{4\pi i\alpha t}}\int_{-\infty}^{\infty}\Psi_0(\xi)\exp\left(-\frac{i(x-\xi)^2}{4i\alpha t}\right)d\xi$$

$$= \frac{1-i}{\sqrt{8\pi\alpha t}}\int_{-\infty}^{\infty}\Psi_0(\xi)\exp\left(-\frac{i(x-\xi)^2}{4i\alpha t}\right)d\xi. \tag{2.12.56}$$

This is the integral solution of the problem. $\square$

Example 2.12.6 (*Slowing Down of Neutrons*).
We consider the problem of slowing down neutrons in an infinite medium with a source of neutrons governed by

$$u_t = u_{xx} + \delta(x)\delta(t), \quad -\infty < x < \infty, \quad t > 0, \tag{2.12.57}$$

$$u(x,0) = \delta(x), \quad -\infty < x < \infty, \tag{2.12.58}$$

$$u(x,t) \to 0 \quad \text{as } |x| \to \infty \text{ for } t > 0, \tag{2.12.59}$$

where $u(x,t)$ represents the number of neutrons per unit volume per unit time, which reach the age t, and $\delta(x)\delta(t)$ is the source function.

Application of the Fourier transform method gives

$$\frac{dU}{dt} + k^2 U = \frac{1}{\sqrt{2\pi}}\,\delta(t),$$
$$U(k,0) = \frac{1}{\sqrt{2\pi}}.$$

The solution of this transformed system is

$$U(k,t) = \frac{1}{\sqrt{2\pi}}\,e^{-k^2 t},$$

and the inverse Fourier transform gives the solution

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2 t}dk = \frac{1}{\sqrt{2\pi}}\mathscr{F}^{-1}\left\{e^{-k^2 t}\right\} \\ &= \frac{1}{\sqrt{4\pi t}}\exp\left(-\frac{x^2}{4t}\right). \end{aligned} \tag{2.12.60}$$

□

Example 2.12.7 (*One-Dimensional Wave Equation*).
Obtain the solution of the one-dimensional wave equation

$$u_{tt} = c^2 u_{xx}, \quad -\infty < x < \infty, \quad t > 0, \tag{2.12.61}$$

$$u(x,0) = 0, \quad u_t(x,0) = \delta(x), \quad -\infty < x < \infty. \tag{2.12.62ab}$$

Making reference to Example 2.12.4, we find $f(x) \equiv 0$ and $g(x) = \delta(x)$ so that $F(k) = 0$ and $G(k) = \frac{1}{\sqrt{2\pi}}$. The solution for $U(k,t)$ is given by

$$U(k,t) = \frac{1}{2c\sqrt{2\pi}}\left[\frac{e^{ickt}}{ik} - \frac{e^{-ickt}}{ik}\right].$$

Thus, the inverse Fourier transform gives

$$\begin{aligned} u(x,t) &= \frac{1}{2c\sqrt{2\pi}}\,\mathscr{F}^{-1}\left\{\frac{e^{ickt}}{ik} - \frac{e^{-ickt}}{ik}\right\} \\ &= \frac{1}{2c\sqrt{2\pi}}\left[\sqrt{\frac{\pi}{2}}\{\mathrm{sgn(x+ct) - sgn(x-ct)}\}\right] \\ &= \frac{1}{4c}[\mathrm{sgn(x+ct) - sgn(x-ct)}] \\ &= \begin{cases} \dfrac{1-1}{4c} = 0, & |x| > ct > 0 \\ \dfrac{1+1}{4c} = \dfrac{1}{2c}, & |x| < ct. \end{cases} \end{aligned}$$

In other words, the solution can be written in the form

$$u(x,t) = \frac{1}{2c} H(c^2t^2 - x^2).$$

□

Example 2.12.8 (*Linearized Shallow Water Equations in a Rotating Ocean*). The horizontal equations of motion of a uniformly rotating inviscid homogeneous ocean of constant depth h are

$$u_t - fv = -g\,\eta_x, \tag{2.12.63}$$

$$v_t + fu = 0, \tag{2.12.64}$$

$$\eta_t + hu_x = 0, \tag{2.12.65}$$

where $f = 2\Omega \sin\theta$ is the Coriolis parameter, which is constant in the present problem, g is the acceleration due to gravity, $\eta(x,t)$ is the free surface elevation, $u(x,t)$ and $v(x,t)$ are the velocity fields. The wave motion is generated by the prescribed free surface elevation at $t = 0$ so that the initial conditions are

$$u(x,0) = 0, \qquad v(x,0) = 0, \qquad \eta(x,0) = \eta_0 H(a - |x|), \tag{2.12.66abc}$$

and the velocity fields and free surface elevation function vanish at infinity.

We apply the Fourier transform with respect to x defined by

$$\mathscr{F}\{f(x,t)\} = F(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x,t)\,dx \tag{2.12.67}$$

to the system (2.12.63)–(2.12.65) so that the system becomes

$$\frac{dU}{dt} - fV = -gikE$$

$$\frac{dV}{dt} + fU = 0$$

$$\frac{dE}{dt} = -hikU$$

$$U(k,0) = 0 = V(k,0), \quad E(k,0) = \sqrt{\frac{2}{\pi}}\,\eta_0 \left(\frac{\sin ak}{k}\right), \tag{2.12.68abc}$$

where $E(k,t) = \mathscr{F}\{\eta(x,t)\}$.

Elimination of U and V from the transformed system gives a single equation for $E(k,t)$ as

$$\frac{d^3E}{dt^3} + \omega^2 \frac{dE}{dt} = 0, \tag{2.12.69}$$

where $\omega^2 = (f^2 + c^2k^2)$ and $c^2 = gh$. The general solution of (2.12.69) is

$$E(k,t) = A + B\cos\omega t + C\sin\omega t, \tag{2.12.70}$$

where A, B, and C are arbitrary constants to be determined from (2.12.68c) and

$$\left(\frac{d^2E}{dt^2}\right)_{t=0} = -c^2k^2E(k,0) = -c^2k^2\cdot\sqrt{\frac{2}{\pi}}\,\eta_0\,\frac{\sin ak}{k},$$

which gives

$$B = \sqrt{\frac{2}{\pi}}\,\eta_0\left(\frac{\sin ak}{k}\right)\cdot\left(\frac{c^2k^2}{\omega^2}\right).$$

Also $\left(\frac{dE}{dt}\right)_{t=0} = 0$ gives $C \equiv 0$ and (2.12.68c) implies $A + B = \sqrt{\frac{2}{\pi}}\eta_0\frac{\sin ak}{k}$. Consequently, the solution (2.12.70) becomes

$$E(k,t) = \sqrt{\frac{2}{\pi}}\,\eta_0\left(\frac{\sin ak}{k}\right)\frac{f^2 + c^2k^2\cos\omega t}{(f^2 + c^2k^2)}. \tag{2.12.71}$$

Similarly

$$U(k,t) = \sqrt{\frac{2}{\pi}}\frac{\eta_0\sin ak}{ih}\cdot\frac{c^2\sin\omega t}{\sqrt{c^2k^2 + f^2}}, \tag{2.12.72}$$

$$V(k,t) = \frac{1}{f}\left(\frac{dU}{dt} + gik\,E\right). \tag{2.12.73}$$

The inverse Fourier transform gives the formal solution for $\eta(x,t)$

$$\eta(x,t) = \left(\frac{\eta_0}{\pi}\right)\int_{-\infty}^{\infty}\frac{\sin ak}{k}\cdot\frac{f^2 + c^2k^2\cos\omega t}{(f^2 + c^2k^2)}e^{ikx}dk. \tag{2.12.74}$$

Similar integral expressions for $u(x,t)$ and $v(x,t)$ can be obtained. □

Example 2.12.9 (*Sound Waves Induced by a Spherical Body*).
We consider propagation of sound waves in an unbounded fluid medium generated by an impulsive radial acceleration of a sphere of radius a. Such waves are assumed to be spherically symmetric and the associated velocity potential on the pressure field $p(r,t)$ satisfies the wave equation

$$\frac{\partial^2 p}{\partial t^2} = c^2\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial p}{\partial r}\right)\right], \tag{2.12.75}$$

where c is the speed of sound. The boundary condition required for the problem is

$$\frac{1}{\rho_0}\left(\frac{\partial p}{\partial r}\right) = -a_0\,\delta(t) \qquad \text{on } r = a, \tag{2.12.76}$$

where ρ_0 is the mean density of the fluid and a_0 is a constant.

Application of the Fourier transform of $p(r,t)$ with respect to time t gives

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dP}{dr}\right)=-k^2P(r,\omega), \tag{2.12.77}$$

$$\frac{dP}{dr}=-\frac{a_0\rho_0}{\sqrt{2\pi}}, \quad \text{on } r=a, \tag{2.12.78}$$

where $\mathscr{F}\{p(r,t)\}=P(r,\omega)$ and $k^2=\frac{\omega^2}{c^2}$.

The general solution of (2.12.77)–(2.12.78) is

$$P(r,\omega)=\frac{A}{r}e^{ikr}+\frac{B}{r}e^{-ikr}, \tag{2.12.79}$$

where A and B are arbitrary constants.

The inverse Fourier transform gives the solution

$$p(r,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{A}{r}e^{i(\omega t+kr)}+\frac{B}{r}e^{i(\omega t-kr)}\right]d\omega. \tag{2.12.80}$$

The first term of the integrand represents incoming spherical waves generated at infinity and the second term corresponds to outgoing spherical waves due to the impulsive radial acceleration of the sphere. Since there is no disturbance at infinity, we impose the *Sommerfeld radiation condition* at infinity to eliminate the incoming waves so that $A=0$, and B is calculated using (2.12.78). Thus, the inverse Fourier transform gives the formal solution

$$p(r,t)=\left(\frac{a_0\rho_0a^2}{2\pi r}\right)\int_{-\infty}^{\infty}\frac{\exp\left[i\omega\left\{t-\frac{r-a}{c}\right\}\right]d\omega}{\left(1+\frac{i\omega a}{c}\right)}. \tag{2.12.81}$$

We next choose a closed contour with a semicircle in the upper half plane and the real ω-axis. Using the Cauchy theory of residues, we calculate the residue contribution from the pole at $\omega=ic/a$. Finally, it turns out that the final solution is

$$u(r,t)=\left(\frac{\rho_0a_0ca}{r}\right)\exp\left[-\frac{c}{a}\left(t-\frac{r-a}{c}\right)\right]H\left(t-\frac{r-a}{c}\right). \tag{2.12.82}$$

$\square$

Example 2.12.10 (*The Linearized Korteweg-de Vries Equation*).
The linearized KdV equation for the free surface elevation $\eta(x,t)$ in an inviscid water of constant depth h is

$$\eta_t+c\eta_x+\frac{ch^2}{6}\eta_{xxx}=0, \quad -\infty<x<\infty, \quad t>0, \tag{2.12.83}$$

where $c = \sqrt{gh}$ is the shallow water speed.

Solve equation (2.12.83) with the initial condition

$$\eta(x,0) = f(x), \quad -\infty < x < \infty. \tag{2.12.84}$$

Application of the Fourier transform $\mathscr{F}\{\eta(x,t)\} = E(k,t)$ to the KdV system gives the solution for $E(k,t)$ in the form

$$E(k,t) = F(k)\exp\left[ikct\left(\frac{k^2h^2}{6} - 1\right)\right].$$

The inverse transform gives

$$\eta(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)\exp\left[ik\left\{(x-ct) + \left(\frac{cth^2}{6}\right)k^2\right\}\right]dk. \tag{2.12.85}$$

In particular, if $f(x) = \delta(x)$, then (2.12.85) reduces to the Airy integral

$$\eta(x,t) = \frac{1}{\pi}\int_0^{\infty} \cos\left[k(x-ct) + \left(\frac{cth^2}{6}\right)k^3\right]dk \tag{2.12.86}$$

which is, in terms of the Airy function,

$$= \left(\frac{cth^2}{2}\right)^{-\frac{1}{3}} Ai\left[\left(\frac{cth^2}{2}\right)^{-\frac{1}{3}}(x-ct)\right], \tag{2.12.87}$$

where the Airy function $Ai(az)$ is defined by

$$Ai(az) = \frac{1}{2\pi a}\int_{-\infty}^{\infty} \exp\left[i\left(kz + \frac{k^3}{3a^3}\right)\right]dk = \frac{1}{\pi a}\int_0^{\infty}\cos\left(kz + \frac{k^3}{3a^3}\right)dk. \tag{2.12.88}$$

$\square$

Example 2.12.11 (*Biharmonic Equation in Fluid Mechanics*).
Usually, the biharmonic equation arises in fluid mechanics and in elasticity. The equation can readily be solved by using the Fourier transform method. We first derive a biharmonic equation from the *Navier-Stokes equations* of motion in a viscous fluid which is given by

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \mathbf{F} - \frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u}, \tag{2.12.89}$$

where $\mathbf{u} = (u, v, w)$ is the velocity field, $\boldsymbol{F}$ is the external force per unit mass of the fluid, p is the pressure, ρ is the density and ν is the kinematic viscosity of the fluid.

The conservation of mass of an incompressible fluid is described by the *continuity equation*

$$\text{div } \mathbf{u} = 0. \tag{2.12.90}$$

In terms of some representative length scale L and velocity scale U, it is convenient to introduce the nondimensional flow variables

$$\mathbf{x}' = \frac{\mathbf{x}}{L}, \quad t' = \frac{Ut}{L}, \quad \mathbf{u}' = \frac{\mathbf{u}}{U}, \quad p' = \frac{p}{\rho U^2}. \tag{2.12.91}$$

In terms of these nondimensional variables, equation (2.12.89) without the external force can be written, dropping the primes, as

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \frac{1}{R}\nabla^2 \mathbf{u}, \tag{2.12.92}$$

where $R = UL/\nu$ is called the *Reynolds number.* Physically, it measures the ratio of inertial forces of the order U^2/L to viscous forces of the order $\nu U/L^2$, and it has special dynamical significance. This is one of the most fundamental nondimensional parameters for the specification of the dynamical state of viscous flow fields.

In the absence of the external force, $\mathbf{F} = \mathbf{0}$, it is preferable to write the Navier-Stokes equations (2.12.89) in the form (since $\mathbf{u} \times \boldsymbol{\omega} = \frac{1}{2}\nabla u^2 - \mathbf{u} \cdot \nabla \mathbf{u}$)

$$\frac{\partial \mathbf{u}}{\partial t} - \mathbf{u} \times \boldsymbol{\omega} = -\nabla\left(\frac{p}{\rho} + \frac{1}{2}u^2\right) - \nu\nabla^2 \mathbf{u}, \tag{2.12.93}$$

where $\boldsymbol{\omega} = \text{curl } \mathbf{u}$ is the *vorticity vector* and $u^2 = \boldsymbol{u} \cdot \boldsymbol{u}$.

We can eliminate the pressure p from (2.12.93) by taking the curl of (2.12.93), giving

$$\frac{\partial \boldsymbol{\omega}}{\partial t} - \text{curl}(\mathbf{u} \times \boldsymbol{\omega}) = \nu\nabla^2 \boldsymbol{\omega} \tag{2.12.94}$$

which becomes, by $\text{div } \mathbf{u} = 0$ and $\text{div } \boldsymbol{\omega} = 0$,

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\boldsymbol{\omega} + \nu\nabla^2 \boldsymbol{\omega}. \tag{2.12.95}$$

This is universally known as the *vorticity transport equation.* The left-hand side represents the rate of change of vorticity. The first two terms on the right-hand side represent the rate of change of vorticity due to stretching and twisting of vortex lines. The last term describes the diffusion of vorticity by molecular viscosity.

In case of two-dimensional flow, $(\boldsymbol{\omega} \cdot \nabla)\mathbf{u} = 0$, equation (2.12.95) becomes

$$\frac{D\boldsymbol{\omega}}{dt} = \frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{u} \cdot \nabla)\boldsymbol{\omega} = \nu\nabla^2 \boldsymbol{\omega}, \tag{2.12.96}$$

where $\mathbf{u} = (u, v, 0)$ and $\boldsymbol{\omega} = (0, 0, \zeta)$, and $\zeta = v_x - u_y$. Equation (2.12.96) shows that only convection and conduction occur. In terms of the stream function $\psi(x, y)$ where

$$u = \psi_y, \quad v = -\psi_x, \quad \boldsymbol{\omega} = -\nabla^2 \psi, \tag{2.12.97}$$

which satisfy (2.12.90) identically, equation (2.12.96) assumes the form

$$\frac{\partial}{\partial t}\left(\nabla^2\psi\right)+\left(\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right)\nabla^2\psi=\nu\nabla^4\psi. \tag{2.12.98}$$

In case of slow motion (velocity is small) or in case of a very viscous fluid (ν very large), the Reynolds number R is very small. For a steady flow in such cases of an incompressible viscous fluid, $\frac{\partial}{\partial t}\equiv 0$, while $(\mathbf{u}\cdot\nabla)\boldsymbol{\omega}$ is negligible in comparison with the viscous term. Consequently, (2.12.98) reduces to the standard *biharmonic equation*

$$\nabla^4\psi=0. \tag{2.12.99}$$

Or, more explicitly,

$$\nabla^2(\nabla^2)\psi\equiv\psi_{xxxx}+2\psi_{xxyy}+\psi_{yyyy}=0. \tag{2.12.100}$$

We solve this equation in a semi-infinite viscous fluid bounded by an infinite horizontal plate at $y=0$, and the fluid is introduced normally with a prescribed velocity through a strip $-a<x<a$ of the plate. Thus, the required boundary conditions are

$$u\equiv\frac{\partial\psi}{\partial y}=0,\quad v\equiv\frac{\partial\psi}{\partial x}=H(a-|x|)f(x)\quad\text{on } y=0, \tag{2.12.101ab}$$

where $f(x)$ is a given function of x.

Furthermore, the fluid is assumed to be at rest at large distances from the plate, that is,

$$(\psi_x,\psi_y)\to(0,0)\quad\text{as } y\to\infty\qquad\text{for } -\infty<x<\infty. \tag{2.12.102}$$

To solve the biharmonic equation (2.12.100) with the boundary conditions (2.12.101ab) and (2.12.102), we introduce the Fourier transform with respect to x

$$\Psi(k,y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ikx}\psi(x,y)dx. \tag{2.12.103}$$

Thus, the Fourier transformed problem is

$$\left(\frac{d^2}{dy^2}-k^2\right)^2\Psi(k,y)=0, \tag{2.12.104}$$

$$\frac{d\Psi}{dy}=0,\quad (ik)\Psi=F(k),\quad y=0, \tag{2.12.105ab}$$

where

$$F(k)=\frac{1}{\sqrt{2\pi}}\int_{-a}^{a}e^{-ikx}f(x)dx. \tag{2.12.106}$$

In view of the Fourier transform of (2.12.102), the bounded solution of (2.12.104) is

$$\Psi(k,y) = (A + B|k|y)\exp(-|k|y), \tag{2.12.107}$$

where A and B can be determined from (2.12.105ab) so that $A = B = (ik)^{-1}F(k)$. Consequently, the solution (2.12.107) becomes

$$\Psi(k,y) = (ik)^{-1}(1 + |k|y)F(k)\exp(-|k|y). \tag{2.12.108}$$

The inverse Fourier transform gives the formal solution

$$\psi(x,y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)G(k)\exp(ikx)dk, \tag{2.12.109}$$

where

$$G(k) = (ik)^{-1}(1 + |k|y)\exp(-|k|y)$$

so that

$$\begin{aligned} g(x) &= \mathscr{F}^{-1}\{G(k)\} = \mathscr{F}^{-1}\{(ik)^{-1}\exp(-|k|y)\} \\ &\qquad\qquad + y\,\mathscr{F}^{-1}\{(ik)^{-1}|k|\exp(-|k|y)\} \\ &= \mathscr{F}_s^{-1}\{k^{-1}\exp(-ky)\} + y\,\mathscr{F}_s^{-1}\{e^{-ky}\}, \end{aligned}$$

which is, by (2.13.7) and (2.13.8),

$$= \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{x}{y}\right) + \sqrt{\frac{2}{\pi}}\frac{xy}{(x^2 + y^2)}. \tag{2.12.110}$$

Using the Convolution Theorem 2.5.5 in (2.12.109) gives the final solution

$$\psi(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x-\xi)\left[\tan^{-1}\left(\frac{\xi}{y}\right) + \frac{y\xi}{\xi^2 + y^2}\right]d\xi. \tag{2.12.111}$$

In particular, if $f(x) = \delta(x)$, then solution (2.12.111) becomes

$$\psi(x,y) = \frac{1}{\pi}\left[\tan^{-1}\left(\frac{x}{y}\right) + \frac{xy}{x^2 + y^2}\right]. \tag{2.12.112}$$

The velocity fields u and v can be determined from (2.12.112). $\square$

Example 2.12.12 (*Biharmonic Equation in Elasticity*).
We derive the biharmonic equation in elasticity from the two-dimensional equilibrium equations and the compatibility condition. In two-dimensional elastic medium, the strain components e_{xx}, e_{xy}, e_{yy} in terms of the displacement functions $(u, v, 0)$ are

$$e_{xx} = \frac{\partial u}{\partial x}, \quad e_{yy} = \frac{\partial v}{\partial y}, \quad e_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right). \tag{2.12.113}$$

Differentiating these results gives the *compatibility condition*

$$\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} = 2\,\frac{\partial^2 e_{xy}}{\partial x \partial y}. \tag{2.12.114}$$

In terms of the *Poisson ratio* ν and *Young's modulus* E of the elastic material, the strain component in the z direction is expressed in terms of stress components

$$Ee_{zz} = \sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy}). \tag{2.12.115}$$

In the case of plane strain, $e_{zz} = 0$, so that

$$\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}). \tag{2.12.116}$$

Substituting this result in other stress-strain relations, we obtain the strain components e_{xx}, e_{xy}, e_{yy} that are related to stress components $\sigma_{xx}, \sigma_{xy}, \sigma_{yy}$ by

$$Ee_{xx} = \sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz}) = (1-\nu^2)\sigma_{xx} - \nu(1+\nu)\sigma_{yy}, \tag{2.12.117}$$
$$Ee_{yy} = \sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz}) = (1-\nu^2)\sigma_{yy} - \nu(1+\nu)\sigma_{xx}, \tag{2.12.118}$$
$$Ee_{xy} = (1+\nu)\sigma_{xy}. \tag{2.12.119}$$

Putting (2.12.117)–(2.12.119) into (2.12.114) gives

$$\frac{\partial^2}{\partial y^2}[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})] + \frac{\partial^2}{\partial x^2}[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})] = 2(1+\nu)\frac{\partial^2 \sigma_{xy}}{\partial x \partial y}. \tag{2.12.120}$$

The basic differential equations for the stress components $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$ in the medium under the action of body forces X and Y are

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \rho X = \rho \frac{\partial^2 u}{\partial t^2}, \tag{2.12.121}$$
$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \rho Y = \rho \frac{\partial^2 \upsilon}{\partial t^2}, \tag{2.12.122}$$

where ρ is the mass density of the elastic material.

The equilibrium equations follow from (2.12.121)–(2.12.122) in the absence of the body forces ($X = Y = 0$) as

$$\frac{\partial}{\partial x}\sigma_{xx} + \frac{\partial}{\partial y}\sigma_{xy} = 0, \tag{2.12.123}$$
$$\frac{\partial}{\partial x}\sigma_{xy} + \frac{\partial}{\partial y}\sigma_{yy} = 0. \tag{2.12.124}$$

It is obvious that the expressions

$$\sigma_{xx} = \frac{\partial^2 \chi}{\partial y^2}, \quad \sigma_{xy} = -\frac{\partial^2 \chi}{\partial x \partial y}, \quad \sigma_{yy} = \frac{\partial^2 \chi}{\partial x^2} \tag{2.12.125}$$

satisfy the equilibrium equations for any arbitrary function $\chi(x,y)$. Substituting from equations (2.12.125) into the compatibility condition (2.12.120), we see that χ must satisfy the *biharmonic equation*

$$\frac{\partial^4\chi}{\partial x^4}+2\,\frac{\partial^4\chi}{\partial x^2\partial y^2}+\frac{\partial^4\chi}{\partial y^4}=0, \tag{2.12.126}$$

which may be written symbolically as

$$\nabla^4\chi=0. \tag{2.12.127}$$

The function χ was first introduced by Airy in 1862 and is known as the *Airy stress function*.

We determine the stress distribution in a semi-infinite elastic medium bounded by an infinite plane at $x=0$ due to an external pressure to its surface. The x-axis is normal to this plane and assumed positive in the direction into the medium. We assume that the external surface pressure p varies along the surface so that the boundary conditions are

$$\sigma_{xx}=-p(y),\quad \sigma_{xy}=0\quad \text{on } x=0\quad \text{for all } y \text{ in } (-\infty,\infty). \tag{2.12.128}$$

We derive solutions so that stress components σ_{xx},σ_{yy}, and σ_{xy} all vanish as $x\to\infty$.

In order to solve the biharmonic equation (2.12.127), we introduce the Fourier transform $\tilde{\chi}(x,k)$ of the *Airy stress function* with respect to y so that (2.12.127)–(2.12.128) reduce to

$$\left(\frac{d^2}{dx^2}-k^2\right)^2\tilde{\chi}=0, \tag{2.12.129}$$

$$k^2\tilde{\chi}(0,k)=\tilde{p}(k),\quad (ik)\left(\frac{d\tilde{\chi}}{dx}\right)_{x=0}=0, \tag{2.12.130}$$

where $\tilde{p}(k)=\mathscr{F}\{p(y)\}$. The bounded solution of the transformed problem is

$$\tilde{\chi}(x,k)=(A+Bx)\exp(-|k|x), \tag{2.12.131}$$

where A and B are constants of integration to be determined from (2.12.130). It turns out that $A=\tilde{p}(k)/k^2$ and $B=\tilde{p}(k)/|k|$ and hence, the solution becomes

$$\tilde{\chi}(x,k)=\frac{\tilde{p}(k)}{k^2}\{1+|k|x\}\exp(-|k|x). \tag{2.12.132}$$

The inverse Fourier transform yields the formal solution

$$\chi(x,y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\tilde{p}(k)}{k^2}(1+|k|x)\exp(iky-|k|x)dk. \tag{2.12.133}$$

The stress components are obtained from (2.12.125) in the form

$$\sigma_{xx}(x,y) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} k^2 \tilde{\chi}(x,k) \exp(iky) dk, \tag{2.12.134}$$

$$\sigma_{xy}(x,y) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (ik) \left(\frac{d\tilde{\chi}}{dx}\right) \exp(iky) dk, \tag{2.12.135}$$

$$\sigma_{yy}(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{d^2\tilde{\chi}}{dx^2} \exp(iky) dk, \tag{2.12.136}$$

where $\tilde{\chi}(x,k)$ are given by (2.12.132). In particular, if $p(y) = P\delta(y)$ so that $\tilde{p}(k) = P(2\pi)^{-\frac{1}{2}}$. Consequently, from (2.12.133)–(2.12.136) we obtain

$$\begin{aligned} \chi(x,y) &= \frac{P}{2\pi} \int_{-\infty}^{\infty} k^{-2}(1+|k|x) \exp(iky - |k|x) dk \\ &= \frac{P}{\pi} \int_0^{\infty} k^{-2}(1+kx) \cos ky \exp(-kx) dk. \end{aligned} \tag{2.12.137}$$

$$\sigma_{xx} = -\frac{P}{\pi} \int_0^{\infty} (1+kx) e^{-kx} \cos ky \, dk = -\frac{2Px^3}{\pi(x^2+y^2)^2}. \tag{2.12.138}$$

$$\sigma_{xy} = -\frac{Px}{\pi} \int_0^{\infty} k \sin ky \exp(-kx) dk = -\frac{2Px^2 y}{\pi(x^2+y^2)^2}. \tag{2.12.139}$$

$$\sigma_{yy} = -\frac{P}{\pi} \int_0^{\infty} (1-kx) \exp(-kx) \cos ky \, dk = -\frac{2Pxy^2}{\pi(x^2+y^2)^2}. \tag{2.12.140}$$

Another physically realistic pressure distribution is

$$p(y) = PH(|a| - y), \tag{2.12.141}$$

where P is a constant, so that

$$\tilde{p}(k) = \sqrt{\frac{2}{\pi}} \frac{P}{k} \sin ak. \tag{2.12.142}$$

Substituting this value for $\tilde{p}(k)$ into (2.12.133)–(2.12.136), we obtain the integral expression for $\chi, \sigma_{xx}, \sigma_{xy}$, and σ_{yy}.

It is noted here that if a point force of magnitude P_0 acts at the origin located on the boundary, then we put $P = (P_0/2a)$ in (2.12.142) and find

$$\tilde{p}(k) = \lim_{a \to 0} \sqrt{\frac{2}{\pi}} \frac{P_0}{2} \left(\frac{\sin ak}{ak}\right) = \frac{P_0}{\sqrt{2\pi}}. \tag{2.12.143}$$

Thus, the stress components can also be written in this case. □

2.13 Fourier Cosine and Sine Transforms with Examples

The Fourier cosine integral formula (2.2.8) leads to the *Fourier cosine transform* and its inverse defined by

$$\mathscr{F}_c\{f(x)\} = F_c(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos kx\, f(x)dx, \tag{2.13.1}$$

$$\mathscr{F}_c^{-1}\{F_c(k)\} = f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos kx\, F_c(k)dk, \tag{2.13.2}$$

where $\mathscr{F}_c$ is the Fourier cosine transform operator and $\mathscr{F}_c^{-1}$ is its inverse operator.

Similarly, the Fourier sine integral formula (2.2.9) leads to the *Fourier sine transform* and its inverse defined by

$$\mathscr{F}_s\{f(x)\} = F_s(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin kx f(x)dx, \tag{2.13.3}$$

$$\mathscr{F}_s^{-1}\{F_s(k)\} = f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin kx\ F_s(k)dk, \tag{2.13.4}$$

where $\mathscr{F}_s$ is the Fourier sine transform operator and $\mathscr{F}_s^{-1}$ is its inverse.

Example 2.13.1 Show that

$$\text{(a)}\ \mathscr{F}_c\{e^{-ax}\} = \sqrt{\frac{2}{\pi}} \frac{a}{(a^2+k^2)}, \quad (a > 0). \tag{2.13.5}$$

$$\text{(b)}\ \mathscr{F}_s\{e^{-ax}\} = \sqrt{\frac{2}{\pi}} \frac{k}{(a^2+k^2)}, \quad (a > 0). \tag{2.13.6}$$

We have

$$\mathscr{F}_c\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos kx\,dx$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty [e^{-(a-ik)x} + e^{-(a+ik)x}]dx$$

$$\mathscr{F}_c\{e^{-ax}\} = \frac{1}{2}\sqrt{\frac{2}{\pi}}\left[\frac{1}{a-ik} + \frac{1}{a+ik}\right] = \sqrt{\frac{2}{\pi}}\frac{a}{(a^2+k^2)}.$$

The proof of the other result is similar and hence left to the reader. ▯

Using the above results with the Fourier cosine and sine inverse transformations and an interchange of variables, we find that

$$\mathscr{F}_c\left\{\frac{1}{(x^2+a^2)}\right\} = \sqrt{\frac{\pi}{2}}\,\frac{e^{-ak}}{a},$$

$$\mathscr{F}_s\left\{\frac{x}{(x^2+a^2)}\right\} = \sqrt{\frac{\pi}{2}}\,e^{-ak}.$$

According to the Fourier cosine and sine inverse transformations, we write

$$e^{-ax} = \frac{2a}{\pi}\int_0^\infty \frac{\cos kx}{k^2+a^2}\,dk = \frac{2}{\pi}\int_0^\infty \frac{k\sin kx}{k^2+a^2}\,dk,\ \ a>0.$$

Interchanging x and k, these results become

$$e^{-ak} = \frac{2a}{\pi}\int_0^\infty \frac{\cos kx}{x^2+a^2}\,dx = \frac{2}{\pi}\int_0^\infty \frac{x\sin kx}{x^2+a^2}\,dx.$$

Thus, it follows that

$$\mathscr{F}_c\left\{\frac{1}{(x^2+a^2)}\right\} = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\cos kx}{x^2+a^2}\,dx = \sqrt{\frac{2}{\pi}}\,\frac{\pi}{2a}\,e^{-ak} = \sqrt{\frac{\pi}{2}}\,\frac{e^{-ak}}{a},$$

$$\mathscr{F}_s\left\{\frac{x}{(x^2+a^2)}\right\} = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{x\sin kx}{x^2+a^2}\,dx = \sqrt{\frac{2}{\pi}}\,\frac{\pi}{2}\,e^{-ak} = \sqrt{\frac{\pi}{2}}\,e^{-ak}.$$

Example 2.13.2 Show that

$$\mathscr{F}_s^{-1}\left\{\frac{1}{k}\exp(-sk)\right\} = \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{x}{s}\right). \tag{2.13.7}$$

We have the standard definite integral

$$\sqrt{\frac{\pi}{2}}\mathscr{F}_s^{-1}\{\exp(-sk)\} = \int_0^\infty \exp(-sk)\sin kx\,dk = \frac{x}{s^2+x^2}. \tag{2.13.8}$$

Integrating both sides with respect to s from s to ∞ gives

$$\begin{aligned}\int_0^\infty \frac{e^{-sk}}{k}\sin kx\,dk &= \int_s^\infty \frac{x ds}{x^2+s^2} = \left[\tan^{-1}\frac{s}{x}\right]_s^\infty \\ &= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{x}\right) = \tan^{-1}\left(\frac{x}{s}\right). \end{aligned}\tag{2.13.9}$$

Thus,

$$\begin{aligned}\mathscr{F}_s^{-1}\left\{\frac{1}{k}\exp(-sk)\right\} &= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{k}\exp(-sk)\sin xk\,dk \\ &= \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{x}{s}\right).\end{aligned}$$

□

Example 2.13.3 Show that

$$\mathscr{F}_s\{erfc(ax)\} = \sqrt{\frac{2}{\pi}}\frac{1}{k}\left[1-\exp\left(-\frac{k^2}{4a^2}\right)\right]. \tag{2.13.10}$$

We have

$$\begin{aligned}\mathscr{F}_s\{erfc(ax)\} &= \sqrt{\frac{2}{\pi}}\int_0^\infty erfc(ax)\sin kx\,dx \\ &= \frac{2\sqrt{2}}{\pi}\int_0^\infty \sin kx\,dx \int_{ax}^\infty e^{-t^2}dt.\end{aligned}$$

Interchanging the order of integration, we obtain

$$\begin{aligned}\mathscr{F}_s\{erf(ax)\} &= \frac{2\sqrt{2}}{\pi}\int_0^\infty \exp(-t^2)dt\int_0^{t/a}\sin kx\,dx \\ &= \frac{2\sqrt{2}}{\pi k}\int_0^\infty \exp(-t^2)\left\{1-\cos\left(\frac{kt}{a}\right)\right\}dt \\ &= \frac{2\sqrt{2}}{\pi k}\left[\frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2}\exp\left(-\frac{k^2}{4a^2}\right)\right].\end{aligned}$$

Thus,

$$\mathscr{F}_s\{erfc(ax)\} = \sqrt{\frac{2}{\pi}}\frac{1}{k}\left[1 - \exp\left(-\frac{k^2}{4a^2}\right)\right].$$

□

2.14 Properties of Fourier Cosine and Sine Transforms

THEOREM 2.14.1 If $\mathscr{F}_c\{f(x)\} = F_c(k)$ and $\mathscr{F}_s\{f(x)\} = F_s(k)$**,** then

$$\mathscr{F}_c\{f(ax)\} = \frac{1}{a}F_c\left(\frac{k}{a}\right), \quad a > 0. \tag{2.14.1}$$

$$\mathscr{F}_s\{f(ax)\} = \frac{1}{a}F_s\left(\frac{k}{a}\right), \quad a > 0. \tag{2.14.2}$$

Under appropriate conditions, the following properties also hold:

$$\mathscr{F}_c\{f'(x)\} = k\ F_s(k) - \sqrt{\frac{2}{\pi}}\, f(0), \tag{2.14.3}$$

$$\mathscr{F}_c\{f''(x)\} = -k^2\ F_c(k) - \sqrt{\frac{2}{\pi}}\, f'(0), \tag{2.14.4}$$

$$\mathscr{F}_s\{f'(x)\} = -k\ F_c(k), \tag{2.14.5}$$

$$\mathscr{F}_s\{f''(x)\} = -k^2\ F_s(k) + \sqrt{\frac{2}{\pi}}\, k\, f(0). \tag{2.14.6}$$

These results can be generalized for the cosine and sine transforms of higher-order derivatives of a function. They are left as exercises.

THEOREM 2.14.2 **(***Convolution Theorem for the Fourier Cosine Transform***).**
If $\mathscr{F}_c\{f(x)\} = F_c(k)$ *and* $\mathscr{F}_c\{g(x)\} = G_c(k)$, then

$$\mathscr{F}_c^{-1}\{F_c(k)G_c(k)\} = \frac{1}{\sqrt{2\pi}}\int_0^\infty f(\xi)[g(x+\xi) + g(|x-\xi|)]d\xi. \tag{2.14.7}$$

Or, equivalently,

$$\int_0^\infty F_c(k)G_c(k)\cos kx\, dk = \frac{1}{2}\int_0^\infty f(\xi)[g(x+\xi) + g(|x-\xi|)]d\xi. \tag{2.14.8}$$

PROOF Using the definition of the inverse Fourier cosine transform, we have

$$\mathscr{F}_c^{-1}\{F_c(k)G_c(k)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty F_c(k)G_c(k)\cos kx\,dk$$
$$= \left(\frac{2}{\pi}\right)\int_0^\infty G_c(k)\cos kx\,dk\int_0^\infty f(\xi)\cos k\xi\,d\xi.$$

Hence,

$$\mathscr{F}_c^{-1}\{F_c(k)G_c(k)\} = \left(\frac{2}{\pi}\right)\int_0^\infty f(\xi)d\xi\int_0^\infty \cos kx\cos k\xi\, G_c(k)dk$$
$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty f(\xi)d\xi\sqrt{\frac{2}{\pi}}\int_0^\infty [\cos k(x+\xi)+\cos k(|x-\xi|)]G_c(k)dk$$
$$= \frac{1}{\sqrt{2\pi}}\int_0^\infty f(\xi)[g(x+\xi)+g(|x-\xi|)]d\xi,$$

in which the definition of the inverse Fourier cosine transform is used. This proves (2.14.7).

It also follows from the proof of Theorem 2.14.2 that

$$\int_0^\infty F_c(k)G_c(k)\cos kx\,dk = \frac{1}{2}\int_0^\infty f(\xi)[g(x+\xi)+g(|x-\xi|)]d\xi.$$

This proves result (2.14.8).

Putting $x=0$ in (2.14.8), we obtain

$$\int_0^\infty F_c(k)G_c(k)dk = \int_0^\infty f(\xi)g(\xi)d\xi = \int_0^\infty f(x)g(x)dx.$$

Substituting $g(x)=\overline{f(x)}$ gives, since $G_c(k)=\overline{F_c(k)}$,

$$\int_0^\infty |F_c(k)|^2dk = \int_0^\infty |f(x)|^2dx. \qquad (2.14.9)$$

This is the *Parseval relation* for the Fourier cosine transform.

Similarly, we obtain

$$\int_0^\infty F_s(k)G_s(k)\cos kx\,dk$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty G_s(k)\cos kx\,dk\int_0^\infty f(\xi)\sin k\xi\,d\xi$$

which is, by interchanging the order of integration,

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty f(\xi)d\xi\int_0^\infty G_s(k)\sin k\xi\cos kx\,dk$$

$$= \frac{1}{2}\int_0^\infty f(\xi)d\xi\sqrt{\frac{2}{\pi}}\int_0^\infty G_s(k)[\sin k(\xi+x)+\sin k(\xi-x)]dk$$

$$= \frac{1}{2}\int_0^\infty f(\xi)[g(\xi+x)+g(\xi-x)]d\xi,$$

in which the inverse Fourier sine transform is used. Thus, we find

$$\int_0^\infty F_s(k)G_s(k)\cos kx\,dk = \frac{1}{2}\int_0^\infty f(\xi)[g(\xi+x)+g(\xi-x)]d\xi. \tag{2.14.10}$$

Or, equivalently,

$$\mathscr{F}_c^{-1}\{F_s(k)G_s(k)\} = \frac{1}{\sqrt{2\pi}}\int_0^\infty f(\xi)[g(\xi+x)+g(\xi-x)]d\xi. \tag{2.14.11}$$

Result (2.14.10) or (2.14.11) is also called the *Convolution Theorem* of the Fourier cosine transform.

Putting $x=0$ in (2.14.10) gives

$$\int_0^\infty F_s(k)G_s(k)dk = \int_0^\infty f(\xi)g(\xi)d\xi = \int_0^\infty f(x)g(x)dx.$$

Replacing $g(x)$ by $\overline{f(x)}$ gives the *Parseval relation* for the Fourier sine transform

$$\int_0^\infty |F_s(k)|^2dk = \int_0^\infty |f(x)|^2dx. \tag{2.14.12}$$

■

2.15 Applications of Fourier Cosine and Sine Transforms to Partial Differential Equations

Example 2.15.1 (*One-Dimensional Diffusion Equation on a Half Line*). Consider the initial-boundary value problem for the one-dimensional diffusion equation in $0 < x < \infty$ with no sources or sinks:

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \infty, \quad t > 0, \tag{2.15.1}$$

where κ is a constant, with the initial condition

$$u(x,0) = 0, \quad 0 < x < \infty, \tag{2.15.2}$$

and the boundary conditions

$$\text{(a) } u(0,t) = f(t), \quad t \geq 0, \quad u(x,t) \to 0 \quad \text{as } x \to \infty, \tag{2.15.3}$$

or,

$$\text{(b) } u_x(0,t) = f(t), \quad t \geq 0, \quad u(x,t) \to 0 \quad \text{as } x \to \infty. \tag{2.15.4}$$

This problem with the boundary conditions (2.15.3) is solved by using the Fourier sine transform

$$U_s(k,t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin kx \, u(x,t) \, dx.$$

Application of the Fourier sine transform gives

$$\frac{dU_s}{dt} = -\kappa k^2 U_s(k,t) + \sqrt{\frac{2}{\pi}} \kappa k f(t), \tag{2.15.5}$$

$$U_s(k,0) = 0. \tag{2.15.6}$$

The bounded solution of this differential system with $U_s(k,0) = 0$ is

$$U_s(k,t) = \sqrt{\frac{2}{\pi}} \kappa k \int_0^t f(\tau) \exp[-\kappa(t-\tau)k^2] \, d\tau. \tag{2.15.7}$$

The inverse transform gives the solution

$$\begin{aligned} u(x,t) &= \sqrt{\frac{2}{\pi}} \kappa \int_0^t f(\tau) \mathscr{F}_s^{-1}\{k \exp[-\kappa(t-\tau)k^2]\} d\tau \\ &= \frac{x}{\sqrt{4\pi\kappa}} \int_0^t f(\tau) \exp\left[-\frac{x^2}{4\kappa(t-\tau)}\right] \frac{d\tau}{(t-\tau)^{3/2}} \end{aligned} \tag{2.15.8}$$

in which $\mathscr{F}_s^{-1}\{k\exp(-t\kappa k^2)\} = \frac{x}{2\sqrt{2}} \cdot \frac{\exp(-x^2/4\kappa t)}{(\kappa t)^{3/2}}$ is used.

In particular, $f(t) = T_0 = \text{constant}$, (2.15.7) reduces to

$$U_s(k,t) = \sqrt{\frac{2}{\pi}}\frac{T_0}{k}[1 - \exp(-\kappa\, t\, k^2)]. \tag{2.15.9}$$

Inversion gives the solution

$$u(x,t) = \left(\frac{2T_0}{\pi}\right)\int_0^\infty \frac{\sin kx}{k}[1 - \exp(-\kappa\, t\, k^2)]dk. \tag{2.15.10}$$

Making use of the integral

$$\int_0^\infty e^{-k^2a^2}\frac{\sin kx}{k}dk = \frac{\pi}{2}\,erf\left(\frac{x}{2a}\right), \tag{2.15.11}$$

the solution becomes

$$\begin{aligned} u(x,t) &= \frac{2T_0}{\pi}\left[\frac{\pi}{2} - \frac{\pi}{2}\,erf\left(\frac{x}{2\sqrt{\kappa t}}\right)\right] \\ &= T_0\; erfc\left(\frac{x}{2\sqrt{\kappa t}}\right), \end{aligned} \tag{2.15.12}$$

where the *error function*, $erf(x)$ is defined by

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-\alpha^2}d\alpha, \tag{2.15.13}$$

so that

$$erf(0) = 0,\; erf(\infty) = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-\alpha^2}d\alpha = 1, \text{ and } erf(-x) = -erf(x),$$

and the *complementary error function*, $erfc(x)$ is defined by

$$erfc(x) = 1 - erf(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-\alpha^2}d\alpha, \tag{2.15.14}$$

so that

$$erfc(x) = 1 - erf(x), \quad erfc(0) = 1, \quad erfc(\infty) = 0,$$

and

$$erfc(-x) = 1 - erf(-x) = 1 + erf(x) = 2 - erfc(x).$$

Equation (2.15.1) with boundary condition (2.15.4) is solved by the Fourier cosine transform

$$U_c(k,t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos kx\, u(x,t)dx.$$

Application of this transform to (2.15.1) gives

$$\frac{dU_c}{dt} + \kappa k^2 U_c = -\sqrt{\frac{2}{\pi}}\,\kappa f(t). \tag{2.15.15}$$

The solution of (2.15.15) with $U_c(k,0) = 0$ is

$$U_c(k,t) = -\sqrt{\frac{2}{\pi}}\,\kappa \int_0^t f(\tau)\exp[-k^2\kappa(t-\tau)]d\tau. \tag{2.15.16}$$

Since

$$\mathscr{F}_c^{-1}\{\exp(-t\kappa k^2)\} = \frac{1}{\sqrt{2\kappa t}}\exp\left(-\frac{x^2}{4\kappa t}\right), \tag{2.15.17}$$

the inverse Fourier cosine transform gives the final form of the solution

$$u(x,t) = -\sqrt{\frac{\kappa}{\pi}} \int_0^t \frac{f(\tau)}{\sqrt{t-\tau}} \exp\left[-\frac{x^2}{4\kappa(t-\tau)}\right] d\tau. \tag{2.15.18}$$

▯

Example 2.15.2 (*The Laplace Equation in the Quarter Plane*).
Solve the Laplace equation

$$u_{xx} + u_{yy} = 0, \quad 0 < x,\ y < \infty, \tag{2.15.19}$$

with the boundary conditions

$$u(0,y) = a, \qquad u(x,0) = 0, \tag{2.15.20a}$$

$$\nabla u \to 0 \qquad \text{as } r = \sqrt{x^2+y^2} \to \infty, \tag{2.15.20b}$$

where a is a constant.

We apply the Fourier sine transform with respect to x to find

$$\frac{d^2U_s}{dy^2} - k^2\, U_s + \sqrt{\frac{2}{\pi}}\, ka = 0.$$

The solution of this inhomogeneous equation is

$$U_s(k,y) = Ae^{-ky} + \sqrt{\frac{2}{\pi}} \cdot \frac{a}{k},$$

where A is a constant to be determined from $U_s(k,0)=0$. Consequently,

$$U_s(k,y)=\frac{a}{k}\sqrt{\frac{2}{\pi}}(1-e^{-ky}). \tag{2.15.21}$$

The inverse transformation gives the formal solution

$$u(x,y)=\frac{2a}{\pi}\int_0^\infty \frac{1}{k}(1-e^{-ky})\sin kx\,dk$$

Or,

$$\begin{aligned} u(x,y) &= \frac{2a}{\pi}\left[\int_0^\infty \frac{\sin kx}{k}dk - \int_0^\infty \frac{1}{k}e^{-ky}\sin kx\,dk\right] \\ &= a-\frac{2a}{\pi}\left(\frac{\pi}{2}-\tan^{-1}\frac{y}{x}\right)=\frac{2a}{\pi}\tan^{-1}\left(\frac{y}{x}\right), \end{aligned} \tag{2.15.22}$$

in which (2.13.9) is used. ▯

Example 2.15.3 (*The Laplace Equation in a Semi-Infinite Strip with the Dirichlet Data*).
Solve the Laplace equation

$$u_{xx}+u_{yy}=0, \quad 0<x<\infty, \quad 0<y<b, \tag{2.15.23}$$

with the boundary conditions

$$u(0,y)=0,\ u(x,y)\to 0 \quad \text{as} \quad x\to\infty \text{ for } 0<y<b \tag{2.15.24}$$

$$u(x,b)=0,\ u(x,0)=f(x) \quad \text{for} \quad 0<x<\infty. \tag{2.15.25}$$

In view of the Dirichlet data, the Fourier sine transform with respect to x can be used to solve this problem. Applying the Fourier sine transform to (2.15.23)–(2.15.25) gives

$$\frac{d^2U_s}{dy^2}-k^2U_s=0, \tag{2.15.26}$$

$$U_s(k,b)=0, \quad U_s(k,0)=F_s(k). \tag{2.15.27}$$

The solution of (2.15.26) with (2.15.27) is

$$U_s(k,y)=F_s(k)\,\frac{\sinh[k(b-y)]}{\sinh kb}. \tag{2.15.28}$$

The inverse Fourier sine transform gives the formal solution

$$\begin{aligned} u(x,y) &= \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(k) \frac{\sinh[k(b-y)]}{\sinh kb} \sin kx \, dk \\ &= \frac{2}{\pi} \int_0^\infty \left[\int_0^\infty f(l) \sin kl \, dl \right] \frac{\sinh[k(b-y)]}{\sinh kb} \sin kx \, dk. \quad (2.15.29) \end{aligned}$$

In the limit as $kb \to \infty$, $\frac{\sinh[k(b-y)]}{\sinh kb} \sim \exp(-ky)$, hence the above problem reduces to the corresponding problem in the quarter plane, $0 < x < \infty, 0 < y < \infty$. Thus, solution (2.15.29) becomes

$$\begin{aligned} u(x,y) &= \frac{2}{\pi} \int_0^\infty f(l) dl \int_0^\infty \sin kl \sin kx \exp(-ky) dk \\ &= \frac{1}{\pi} \int_0^\infty f(l) dl \int_0^\infty \{\cos k(x-l) - \cos k(x+l)\} \exp(-ky) dk \\ &= \frac{1}{\pi} \int_0^\infty f(l) \left[\frac{y}{(x-l)^2 + y^2} - \frac{y}{(x+l)^2 + y^2} \right] dl. \quad (2.15.30) \end{aligned}$$

This is the exact integral solution of the problem. If $f(x)$ is an odd function of x, then solution (2.15.30) reduces to the solution (2.12.10) of the same problem in the half plane. □

2.16 Evaluation of Definite Integrals

The Fourier transform can be employed to evaluate certain definite integrals. Although the method of evaluation may not be very rigorous, it is quite simple and straightforward. The method can be illustrated by means of examples.

Example 2.16.1 Evaluate the integral

$$I(a,b) = \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}, \quad a > 0, b > 0. \quad (2.16.1)$$

If we write $f(x) = \frac{1}{x^2+a^2}$ and $g(x) = \frac{1}{x^2+b^2}$ so that $F(k) = \sqrt{\frac{\pi}{2}} \frac{e^{-a|k|}}{a}$, $G(k) =$

$\sqrt{\frac{\pi}{2}}\frac{e^{-b|k|}}{b}$ and use the formula (2.5.19), we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)g(-x)dx &= \int_{-\infty}^{\infty} F(k)G(k)dk \\ &= \frac{\pi}{2ab}\int_{-\infty}^{\infty} e^{-|k|(a+b)}dk \\ &= \frac{\pi}{ab}\int_{0}^{\infty} e^{-(a+b)k}dk = \frac{\pi}{ab(a+b)}. \end{aligned} \tag{2.16.2}$$

This is the desired result. Further

$$\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2ab(a+b)}. \tag{2.16.3}$$

□

Example 2.16.2 Show that

$$\int_0^\infty \frac{x^{-p}dx}{(a^2+x^2)} = \frac{\pi}{2}a^{-(p+1)}\sec\left(\frac{\pi p}{2}\right). \tag{2.16.4}$$

We write

$$f(x) = e^{-ax} \quad \text{so that} \quad F_c(k) = \sqrt{\frac{2}{\pi}}\frac{a}{(a^2+k^2)}.$$

$$g(x) = x^{p-1} \quad \text{so that} \quad G_c(k) = \sqrt{\frac{2}{\pi}}k^{-p}\Gamma(p)\cos\left(\frac{\pi p}{2}\right).$$

Using Parseval's result for the Fourier cosine transform gives

$$\int_0^\infty F_c(k)G_c(k)dk = \int_0^\infty f(x)g(x)dx.$$

Or,

$$\begin{aligned}\frac{2a}{\pi}\cos\left(\frac{\pi p}{2}\right)\Gamma(p)\int_0^\infty \frac{k^{-p}dk}{k^2+a^2} &= \int_0^\infty x^{p-1}e^{-ax}dx \\ &= \frac{1}{a^p}\int_0^\infty e^{-t}t^{p-1}dt = \frac{\Gamma(p)}{a^p}, \quad (ax=t).\end{aligned}$$

Thus,

$$\int_0^\infty \frac{k^{-p}dk}{a^2+k^2} = \frac{\pi}{2\,a^{p+1}} \sec\left(\frac{\pi p}{2}\right).$$

▯

Example 2.16.3 If $a>0, b>0$, show that

$$\int_0^\infty \frac{x^2\,dx}{(a^2+x^2)(b^2+x^2)} = \frac{\pi}{2(a+b)}. \tag{2.16.5}$$

We consider

$$\mathscr{F}_s\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}\frac{k}{k^2+a^2} = F_s(k)$$
$$\mathscr{F}_s\{e^{-bx}\} = \sqrt{\frac{2}{\pi}}\frac{k}{k^2+b^2} = G_s(k).$$

Then the Convolution Theorem for the Fourier cosine transform gives

$$\int_0^\infty F_s(k)G_s(k)\cos kx\,dk = \frac{1}{2}\int_0^\infty g(\xi)[f(\xi+x)+f(\xi-x)]d\xi.$$

Putting $x=0$ gives

$$\int_0^\infty F_s(k)G_s(k)dk = \int_0^\infty g(\xi)f(\xi)d\xi,$$

or,

$$\int_0^\infty \frac{k^2\,dk}{(k^2+a^2)(k^2+b^2)} = \frac{\pi}{2}\int_0^\infty e^{-(a+b)\xi}d\xi = \frac{\pi}{2(a+b)}.$$

▯

Example 2.16.4 Show that

$$\int_0^\infty \frac{x^2dx}{(x^2+a^2)^4} = \frac{\pi}{(2a)^5}, \quad a>0. \tag{2.16.6}$$

We write $f(x) = \frac{1}{2(x^2+a^2)}$ so that $f'(x) = -\frac{x}{(x^2+a^2)^2}$, and $\mathscr{F}\{f(x)\} = F(k) = \sqrt{\frac{\pi}{2}}\left(\frac{1}{2a}\right)\exp(-a|k|)$.

Making reference to the Parseval relation (2.4.19), we obtain

$$\int_{-\infty}^{\infty} |f'(x)|^2 dx = \int_{-\infty}^{\infty} |\mathscr{F}\{f'(x)\}|^2 dk = \int_{-\infty}^{\infty} |(ik)\mathscr{F}\{f(x)\}|^2 dk.$$

Thus,

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)^4}\,dx = \frac{\pi}{2}\int_{-\infty}^{\infty} k^2 \cdot \frac{1}{(2a)^2}\exp(-2a|k|)dk$$
$$= \frac{\pi}{(2a)^2}\int_0^{\infty} k^2 \exp(-2ak)dk = \frac{2\pi}{(2a)^5}.$$

This gives the desired result. ▯

Example 2.16.5 Show that

$$\int_{-\infty}^{\infty} e^{-(a+b)x^2} dx = \sqrt{\frac{\pi}{a+b}}, \quad a>0,\ b>0. \tag{2.16.7}$$

We write $f(x)=e^{-ax^2}$ and $g(x)=e^{-bx^2}$ so that $F(k)=\frac{1}{\sqrt{2a}}\,e^{-\frac{k^2}{4a}}$, and $G(k)=\frac{1}{\sqrt{2b}}\,e^{-\frac{k^2}{4b}}$ and then use the formula (2.5.19) to obtain

$$\int_{-\infty}^{\infty} f(x)g(-x)dx = \frac{1}{2\sqrt{ab}}\int_{-\infty}^{\infty} e^{-\frac{k^2}{4}(\frac{1}{a}+\frac{1}{b})}dk$$
$$= \frac{1}{2\sqrt{ab}}\int_{-\infty}^{\infty} e^{-ck^2}dk, \quad c=\frac{1}{4}(\frac{1}{a}+\frac{1}{b})$$
$$= \frac{1}{2\sqrt{ab}}\sqrt{\frac{\pi}{c}} = \sqrt{\frac{\pi}{a+b}}.$$

▯

2.17 Applications of Fourier Transforms in Mathematical Statistics

In probability theory and mathematical statistics, the characteristic function of a random variable is defined by the Fourier transform or by the Fourier-Stieltjes transform of the distribution function of a random variable. Many

INTEGRAL TRANSFORMS AND THEIR APPLICATIONS

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