Department of Physics
Taki Government College

STUDY MATERIAL

FOR SEM-4 CORE

NUCLEAR PHYSICS

Sources of the Materials:

1.http://www.brainkart.com/article/Classification-of-nuclei-and-Properties-of-nucleus-Nuclear-size_2954/

2. https://www.soton.ac.uk/~ab1u06//teaching/phys3002/notes.html

3. https://ocw.mit.edu/courses/nuclear-engineering/22-02-introduction-to-applied-nuclear-physics-spring-2012/lecture-notes/MIT22_02S12_lec_ch7.pdf

Nucleus

The nucleus consists of the elementary particles, protons and neutrons which are known as nucleons. A proton has positive charge of the same magnitude as that of electron and its rest mass is about 1836 times the mass of an electron. A neutron is electrically neutral, whose mass is almost equal to the mass of the proton. The nucleons inside the nucleus are held together by strong attractive forces called nuclear forces.

A nucleus of an element is represented as ${}_{Z}X^{A}$, where X is the chemical symbol of the element. Z represents the atomic number which is equal to the number of protons and A, the mass number which is equal to the total number of protons and neutrons. The number of neutrons is represented as N which is equal to A−Z. For example, the chlorine nucleus is represented as ${}_{17}Cl^{35}$. It contains 17 protons and 18 neutrons.

1Classification of nuclei

(i) Isotopes

Isotopes are atoms of the same element having the same atomic number Z but different mass number A. The nuclei ${}_{1}H^{1}$, ${}_{1}H^{2}$ and ${}_{1}H^{3}$ are the isotopes of hydrogen. In other words isotopes of an element contain the same number of protons but different number of neutrons. As the atoms of isotopes have identical electronic structure, they have identical chemical properties and placed in the same location in the periodic table.

(ii) Isobars

Isobars are atoms of different elements having the same mass number A, but different atomic number Z. The nuclei ${}_{8}O^{16}$ and ${}_{7}N^{16}$ represent two isobars. Since isobars are atoms of different elements, they have different physical and chemical properties.

(iii) Isotones

Isotones are atoms of different elements having the same number of neutrons. ${}_{6}C^{14}$ and ${}_{8}O^{16}$ are some examples of isotones.

2.General properties of nucleus
Nuclear size

According to Rutherford's α−particle scattering experiment, the distance of the closest approach of α − particle to the nucleus was taken as a measure of nuclear radius, which is approximately 10^{-15} m. If the nucleus is assumed to be spherical, an empirical relation is found to hold good between the radius of the nucleus R and its mass number A. It is given by

$$R \propto A^{1/3}$$

$$R = r_o A^{1/3}$$

where r_o is the constant of proportionality and is equal to 1.3 F (1 Fermi, F = 10^{-15} m)

Nuclear density

The nuclear density ρ_N can be calculated from the mass and size of the nucleus.

ρ_N = Nuclear mass / Nuclear volume

Nuclear mass = Am_N

where, A = mass number

and m_N = mass of one nucleon and is approximately equal to 1.67 x 10^{-27} kg

Nuclear volume = $4/3\ \pi R^3$

$\rho_N = m_N / (4/3\ \pi r_0^3)$

Substituting the known values, the nuclear density is calculated as 1.816 x 10^{17} kg m^{-3} which is almost a constant for all the nuclei irrespective of its size.

The high value of the nuclear density shows that the nuclear matter is in an extremely compressed state.

Nuclear charge

The charge of a nucleus is due to the protons present in it. Each proton has a positive charge equal to 1.6 x 10^{-19} C.

The nuclear charge = Ze, where Z is the atomic number.

Atomic mass unit

It is convenient to express the mass of a nucleus in atomic mass unit (amu), though the unit of mass is kg. One atomic mass unit is considered as one twelfth of the mass of carbon atom $_6C^{12}$. Carbon of atomic number 6 and mass number 12 has mass equal to 12 amu.

1 amu = 1.66×10^{-27} kg

The mass of a proton, m_p = 1.007276 amu

This is equal to the difference in mass of the hydrogen atom which is 1.007825 amu and the mass of electron.

The mass of a neutron, m_n = 1.008665 amu

The energy equivalence of one amu can be calculated in electron-volt

Einstein's mass energy relation is, $E = mc^2$ Here, m = 1 amu = 1.66×10^{-27} kg

$c = 3 \times 10^8$ ms^{-1}

$E = 1.66 \times 10^{-27} \times (3 \times 10^8)^2$ J

One electron-volt (eV) is defined as the energy of an electron when it is accelerated through a potential difference of 1 volt.

1 eV = 1.6×10^{-19} coulomb x 1 volt, 1 eV = 1.6×10^{-19} joule

Hence, $E = 1.66 \times 10^{-27} \times (3 \times 10^8)^2 / 1.6 \times 10^{-19}$ eV

= 931 million electronvolt = 931 MeV

Thus, energy equivalent of 1 amu = 931 MeV

Nuclear mass

As the nucleus contains protons and neutrons, the mass of the nucleus is assumed to be the mass of its constituents.

Assumed nuclear mass = $Zm_P + Nm_n$,

where m_p and m_n are the mass of a proton and a neutron respectively. However, from the measurement of mass by mass spectrometers, it is found that the mass of a stable nucleus (m) is less than the total mass of the nucleons.

i.e mass of a nucleus, $m < (Zm_p + Nm_n)$ $Zm_p + Nm_N - m = \Delta m$

where Δm is the mass defect

Thus, the difference in the total mass of the nucleons and the actual mass of the nucleus is known as the mass defect.

Note : In any mass spectrometer, it is possible to determine only the mass of the atom, which includes the mass of Z electrons. If M represents the mass of the atom, then the mass defect can be written as

$\Delta m = Zm_P + Nm_n + Zm_e - M$

$= Zm_H + Nm_n - M$

where m_H represents the mass of one hydrogen atom

Binding energy

When the protons and neutrons combine to form a nucleus, the mass that disappears (mass defect, Δm) is converted into an equivalent amount of energy (Δmc^2). This energy is called the binding energy of the nucleus.

Binding energy $= [Zm_P + Nm_n - m]\ c^2$

$= \Delta m\ c^2$

The binding energy of a nucleus determines its stability against disintegration. In other words, if the binding energy is large, the nucleus is stable and vice versa.

The binding energy per nucleon is

BE/ A = Binding energy of the nucleus / Total number of nucleons

It is found that the binding energy per nucleon varies from element to element. A graph is plotted with the mass number A of the nucleus along the X−axis and binding energy per nucleon along the Y-axis (Fig).

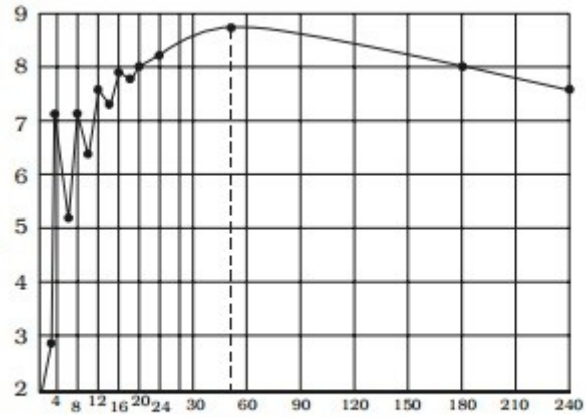


Fig Binding energy graph

Explanation of binding energy curve

i. The binding energy per nucleon increases sharply with mass number A upto 20. It increases slowly after A = 20. For A<20, there exists recurrence of peaks corresponding to those nuclei, whose mass numbers are multiples of four and they contain not only equal but also even number of protons and neutrons. Example: $_2He^4$, $_4Be^8$, $_6C^{12}$, $_8O^{16}$, and $_{10}Ne^{20}$. The curve becomes almost flat for mass number between 40 and 120. Beyond 120, it decreases slowly as A increases.

ii. The binding energy per nucleon reaches a maximum of MeV at A=56, corresponding to the iron nucleus ($_{26}Fe^{56}$). Hence, iron nucleus is the most stable.

iii. The average binding energy per nucleon is about 8.5 MeV for nuclei having mass number ranging between 40 and 120. These elements are comparatively more stable and non radioactive.

iv. For higher mass numbers the curve drops slowly and the BE/A is about 7.6 MeV for uranium. Hence, they are unstable and radioactive.

Chapter 7

Alpha Decay

α- decay is the radioactive emission of an α-particle which is the nucleus of ${}^4_2\mathrm{He}$, consisting of two protons and two neutrons. This is a very stable nucleus as it is doubly magic. The daughter nucleus has two protons and four nucleons fewer than the parent nucleus.

$$ {}^{(A+4)}_{(Z+2)}\{P\} \;\rightarrow\; {}^{A}_{Z}\{D\} \;+\; \alpha. $$

7.1 Kinematics

The "Q-value" of the decay, Q_α is the difference of the mass of the parent and the combined mass of the daughter and the α-particle, multiplied by c^2.

$$ Q_\alpha \;=\; (m_P \;-\; m_D \;-\; m_\alpha)\, c^2. $$

The mass difference between the parent and daughter nucleus can usually be estimated quite well from the Liquid Drop Model. It is also equal to the difference between the sum of the binding energies of the daughter and the α-particles and that of the parent nucleus.

The α-particle emerges with a kinetic energy T_α, which is slightly below the value of Q_α. This is because if the parent nucleus is at rest before decay there must be some recoil of the daughter nucleus in order to conserve momentum. The daughter nucleus therefore has kinetic energy T_D such that

$$ Q_\alpha \;=\; T_\alpha \;+\; T_D $$

The momenta of the α-particle and daughter nucleus are respectively

$$ p_\alpha \;=\; \sqrt{2m_\alpha T_\alpha}, $$

$$ p_D \;=\; -\sqrt{2m_D T_D}, $$

where m_D is the mass of the daughter nucleus (we have taken the momentum of the α-particle to be positive). Conserving momentum implies $p_\alpha + p_D = 0$ which leads to

$$ T_D = \frac{m_\alpha}{m_D} T_\alpha, $$

and neglecting the binding energies, we have

$$\frac{m_\alpha}{m_D} \;=\; \frac{4}{A},$$

where A is the atomic mass number of the daughter nucleus. We therefore have for the kinetic energy of the α-particle

$$T_\alpha \;=\; \frac{A}{(A+4)} Q_\alpha.$$

Example:
The binding energy of $^{214}_{84}$Po is 1.66601 GeV, the binding energy of $^{210}_{82}$Pb (lead) is 1.64555 GeV and the binding energy of $^{4}_{2}$He is 28.296 MeV. The Q-value for the decay

$$^{214}_{84}\mathrm{Po} \;\rightarrow\; ^{210}_{82}\mathrm{Pb} \;+\; \alpha,$$

is therefore

$$Q_\alpha \;=\; 1645.55 + 28.296 - 1666.02 \;=\; 7.83\,\mathrm{MeV}.$$

The kinetic energy of the α-particle is then given by

$$T_\alpha \;=\; \frac{210}{214} \times 7.83 \;=\; 7.68\,\mathrm{MeV}.$$

Sometimes the α-particles emerge with kinetic energies which are somewhat lower than this prediction. Such α-decays are accompanied by the emission of γ-rays. What is happening is that the daughter nucleus is being produced in one of its excited states, so that there is less energy available for the α-particle (or the recoil of the daughter nucleus).

Example:
The binding energy of $^{228}_{90}$Th (thorium) is 1.743077 GeV, the binding energy of $^{224}_{88}$Ra (radium) is 1.720301 GeV and the binding energy of $^{4}_{2}$He is 28.296 MeV. The Q-value for the decay

$$^{228}_{90}\mathrm{Th} \;\rightarrow\; ^{224}_{88}\mathrm{Ra} \;+\; \alpha,$$

is therefore

$$Q_\alpha \;=\; 1720.301 + 28.296 - 1743.077.02 \;=\; 5.52\,\mathrm{MeV}.$$

The kinetic energy of the α-particle is then given by

$$T_\alpha \;=\; \frac{224}{228} \times 5.52 \;=\; 5.42\,\mathrm{MeV}.$$

$\alpha-$particles are observed with this kinetic energy, but also with kinetic energies 5.34, 5.21, 5.17 and 5.14 MeV.

From this we can conclude that there are excited states of $^{224}_{88}$Ra with energies of 0.08, 0.21, 0.25 and 0.28 MeV. The α-decay is therefore accompanied by γ-rays (photons) with energies equal to the differences of these energies.

It is sometimes possible to find an α-particle whose energy is *larger* than that predicted from the Q-value. This occurs when the parent nucleus is itself a product of a decay from a

further ('grand'-)parent. In this case the parent α-decaying nucleus can be produced in one of its excited states. In most cases this state will decay to the ground state by emitting γ-rays before the α-decay takes places. But in some cases where the excited state is relatively long-lived and the decay constant for the α-decay is large the excited state can α-decay directly and the Q-value for such a decay is larger than for decay form the ground state by an amount equal to the excitation energy.

In the above example of α-decay from $^{214}_{84}$Po (polonium) the parent nucleus is actually unstable and is produced by β-decay of $^{214}_{83}$Bi (bismuth). $^{214}_{84}$Po has excited states with energies 0.61, 1.41, 1.54, 1.66 MeV above the gound state. Therefore as well as an α-decay with Q-value 7.83 MeV, calculated above, there are α-decays with Q-values of 8.44, 9.24, 9.37 and 9.49 MeV.

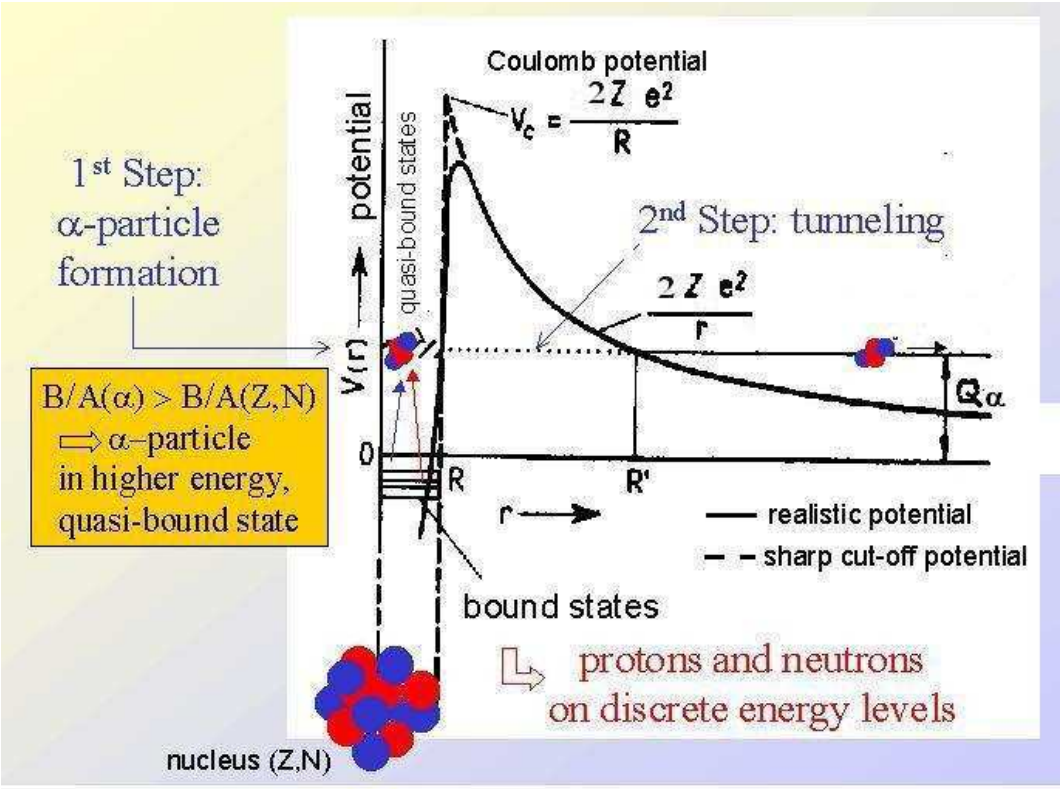
7.2 Decay Mechanism

The mean lifetime of α-decaying nuclei varies from the order of 10^{-7} secs to 10^{10} years.

We can understand this by investigating the mechanism for α-decay.

What happens is that two protons from the highest proton energy levels and two neutrons from the highest neutron energy levels combine to form an α-particle inside the nucleus - this is known as a "quasi-bound-state". It acquires an energy which is approximately equal to Q_α (we henceforth neglect the small correction due to the recoil of the nucleus).

The α-particle is bound to the potential well created by the strong, short-range, nuclear forces. There is also a Coulomb repulsion between this 'quasi-' α-particle and the rest of the nucleus.



Together these form a potential barrier, whose height, V_c, is the value of the Coulomb potential at the radius, R, of the nucleus (where the strong interactions are rapidly attenuated).

$$V_c \ = \ \frac{2Ze^2}{4\pi\epsilon_0 R},$$

where Ze is the electric charge of the daughter nucleus.

The barrier extends from $r = R$, the nuclear radius to $r = R'$, where

$$Q_\alpha \ = \ \frac{2Ze^2}{4\pi\epsilon_0 R'}.$$

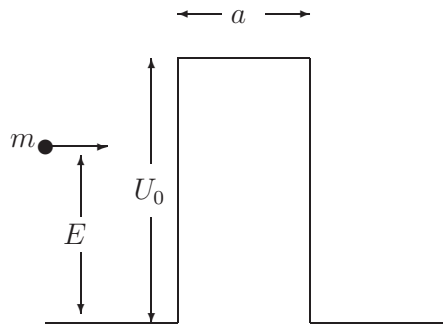
Beyond R' the α-particle has enough energy to escape.

Using classical mechanics, the α-particle does not have enough energy to cross this barrier, but it *can* penetrate through via quantum tunnelling.

For a square potential of height U_0 and width a, the tunnelling probability for a particle with mass, m and energy E, is approximately given by

$$T \ = \ \exp\left(-2\sqrt{2m(U_0 - E)}\frac{a}{\hbar}\right).$$

It is this exponential which varies very rapidly with its argument, that is responsible for the huge variation in α-decay constants.



This formula applies to a potential barrier of constant height U_0, whereas for α-decay the potential inside the barrier is

$$U(r) \;=\; \frac{2Ze^2}{4\pi\epsilon_0 r}.$$

The result of this is that the exponent in the above expression is replaced by the integral

$$-\frac{2}{\hbar}\int_R^{R'} \sqrt{2m_\alpha\left(\frac{2Ze^2}{4\pi\epsilon_0 r} - Q_\alpha\right)}\, dr$$

Finally we need to multiply the transition probability by the number of times per sec that the α-particle 'tries' to escape, which is how often it can travel from the centre to the edge of the nucleus and back. This is approximately given by

$$\frac{v}{2R},$$

where $v = \sqrt{2Q_\alpha/m_\alpha}$, is the velocity of the α-particle inside the nucleus.

When all this is done we arrive at the approximate result

$$\ln\lambda \;=\; f \;-\; g\,\frac{Z}{\sqrt{Q_\alpha}},$$

where

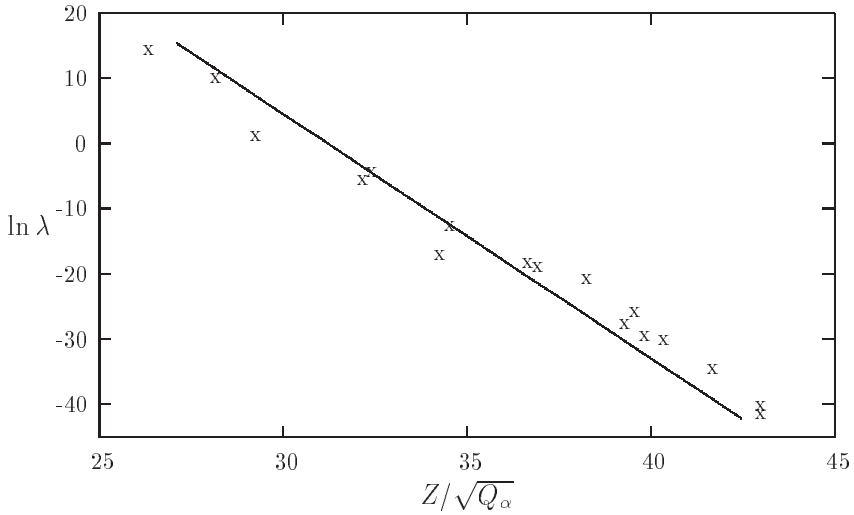
$$g \;=\; 2\sqrt{2}\pi\alpha\sqrt{m_\alpha\, c^2} \;=\; 3.97\,\mathrm{MeV}^{1/2},$$

and

$$f \;=\; \ln\left(\frac{v}{2R}\right) \;+\; 8\sqrt{RZ\alpha m_\alpha c/\hbar}.$$

f varies somewhat for different nuclei but is approximately equal to 128.

This very crude approximation agrees reasonably well with data



We see that as the quantity $Z/\sqrt{Q_\alpha}$ varies over the range 25 - 45, the logarithm of the decay constant varies over a similar range from -45 to 15, but this implies a range of lifetimes from e^{-15} to e^{45} secs (less than a microsecond to longer than the age of the Universe)

Chapter 8

Beta Decay

β-decay is the radioactive decay of a nuclide in which an electron or a positron is emitted.

$$ {}^{A}_{Z}\{P\} \;\rightarrow\; {}^{A}_{(Z+1)}\{D\} \;+\; e^{-} \;+ \bar{\nu}, $$

or

$$ {}^{A}_{Z}\{P\} \;\rightarrow\; {}^{A}_{(Z-1)}\{D\} \;+\; e^{+} \;+ \nu. $$

The atomic mass number is unchanged so that these reactions occur between "isobars".

The electron (or positron) does not exist inside the nucleus but is created in the reaction

$$ n \;\rightarrow\; p \;+\; e^{-} \;+ \bar{\nu}. $$

In fact the neutron has a mass that exceeds the sum of the masses of the proton plus the electron so that a free neutron can undergo this decay with a lifetime of about 11 minutes.

Inside a nucleus such a decay is not always energetically allowed because of the difference in the binding energies of the parent and daughter nuclei. When a neutron is converted into a proton the Coulomb repulsion between the nucleons increases - thereby decreasing the binding energy. Moreover there is a pairing term in the semi-empirical mass formula that favours even numbers of protons and neutrons and a symmetry term that tells us that the number of protons and neutrons should be roughly equal.

β-decay is energetically permitted provided the mass of the parent exceeds the mass of the daughter plus the mass of an electron.

$$ M(Z, A) \;>\; M((Z+1), A) \;+\; m_e, $$

for electron emission, and

$$ M(Z, A) \;>\; M((Z-1), A) \;+\; m_e, $$

for positron emission. In the latter case a proton is converted into a more massive neutron, but the binding energy of the daughter may be such that the total nuclear mass of the daughter is less than that of the parent by more than the electron mass, m_e.

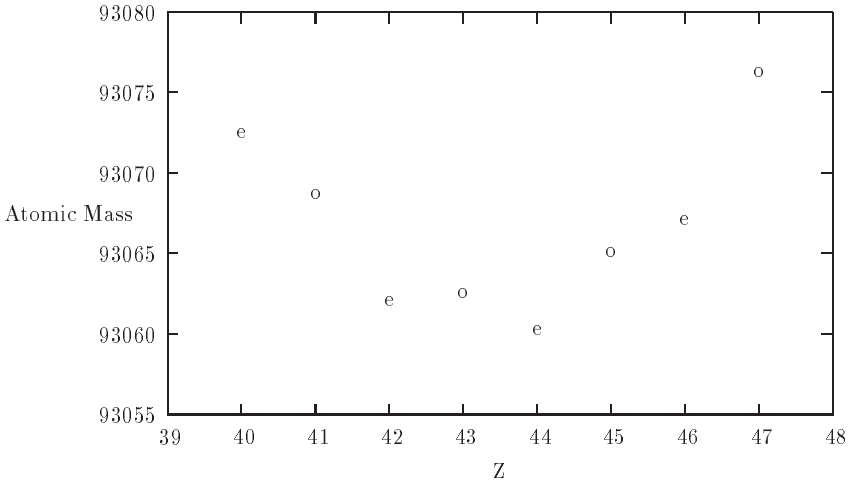
The mass of the electron can be included directly by comparing atomic masses, since a neutral atom always has Z electrons. Thus we require

$$\mathcal{M}(Z, A) \; > \; \mathcal{M}((Z+1), A)$$

for electron emission. The atomic (as opposed to nuclear) mass included the masses of the electrons. However, this will not work for positron emission, for which Z *decreases* by one unit.

For nuclei with even A, it turns out that because of the pairing term in the binding energy, nuclides with odd numbers of protons and neutrons (odd-odd nuclides) are nearly always unstable against β- decay. On the other hand, even-even nuclides can also sometimes be unstable against β-decay if the number of neutrons in a particular isobar is too large or too small for stability.

For example, consider the isobars for A=100.



We note that all the odd-odd nuclides marked "o" have a larger atomic mass than one of the adjacent even-even (marked "e") nuclides and that for the case of Z=43, *both* electron and positron emission are energetically allowed so that this nuclide (Tc – Technetium) can decay either by electron emission to Z=44 (Ru – Ruthenium) or by positron emission to Z=42 (Mo – Molibdenium). Moreover, the even-even Z=40 nuclide (Zr – Zirconium) can decay by electron emission to Z=41 (Nb – Niobium).

For nuclei with odd A there is either an even number of neutrons *or* an even number of protons. In this case the pairing term does not change from isobar to isobar and the question of stability relies on the balance between the symmetry term which prefers equal numbers of protons and neutrons and the Coulomb terms which prefers fewer protons. For such nuclides there is only one stable isobar, with some atomic number Z_A. This means that the isobars with atomic number $Z > Z_A$ have too many protons for stability can always β-decay emitting a positron, whereas isobars with $Z < Z_A$ have too many neutrons, and can undergo β-decay emitting an electron. The value of Z_A for a given A can be obtained by minimizing the *atomic* mass (including the masses of the electrons) from the semi-empirical

mass formula. This gives

$$Z_A \;=\; A\frac{2a_A + (m_n - m_p - m_e)c^2/2}{4a_A + a_C A^{2/3}},$$

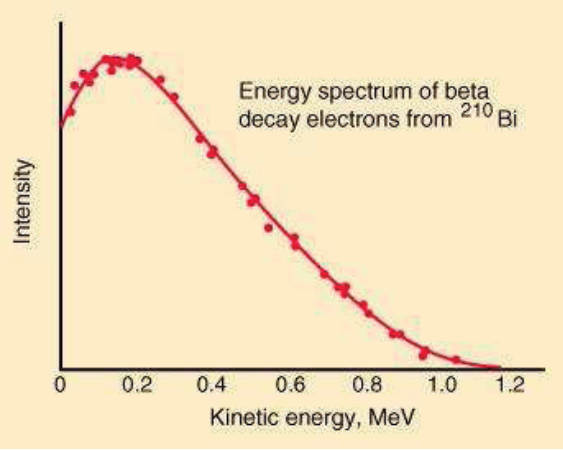
where a_A and a_C are the coefficients of the asymmetry term and Coulomb term in the semi-empirical mass formula.

8.1 Neutrinos

As in the case of α-decay the difference between the mass of the parent nucleus, m_P and the mass of the daughter, m_D plus the electron is the Q-value for the decay, Q_β,

$$Q_\beta \;=\; (m_P - m_D - m_e)c^2,$$

and in this case the recoil of the daughter can be neglected because the electron is so much lighter than the nuclei. We would expect this Q-value to be equal to the kinetic energy of the emitted electron, but what is observed is a spectrum of electron energies up to a maximum value which is equal to this Q-value. For example the intensity of electrons with different energies form the β-decay of $^{210}_{83}$Bi (bismuth) is



There is a further puzzle. Since the number of spin-$\frac{1}{2}$ nucleons is the same in the parent and daughter nuclei, the difference in the spins of the parent and daughter nuclei must be an integer. But the electron also has spin-$\frac{1}{2}$, so there appears to be a violation of conservation of angular momentum here.

The solution to both of these puzzles was provided in 1930 by Pauli who postulated the existence of a massless neutral particle with spin-$\frac{1}{2}$ which always accompanies the electron in β-decay. This was called a neutrino. Neutrinos interact very weakly with matter and so they were not actually detected until 1953 (by Reines and Cowan). The fact that the neutrino has spin-$\frac{1}{2}$ means that the total angular momentum can be conserved (if necessary the electron-antineutrino system has orbital angular momentum) and the Q-value is the sum of the energies of the electron and antineutrino. The kinetic energy of the electron can vary

from zero (strictly arbitrarily small) where all the Q-value is taken by the antineutrino (the momentum being conserved by the small recoil of the daughter nucleus) to the Q-value in which case the energy carried off by the antineutrino is negligible.

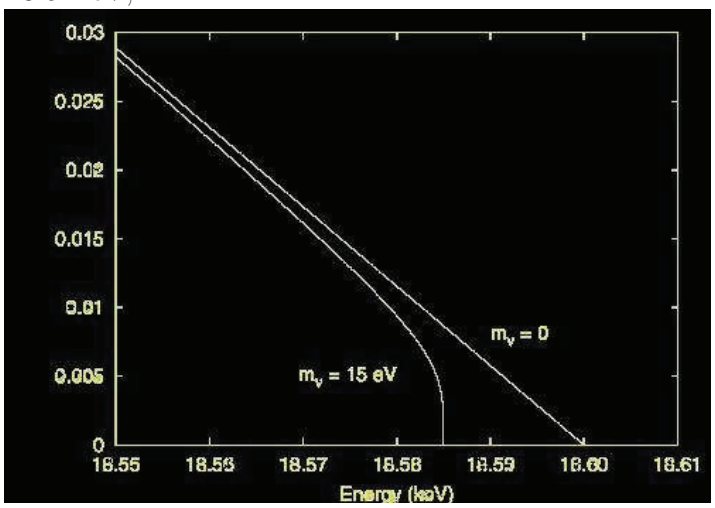
Electrons and neutrinos are examples of "leptons" which are particles that do not interact under the strong nuclear forces - they are not found inside nuclei.

By convention, electrons and neutrinos are assigned a "lepton number" of 1, which means that positrons and antineutrinos have a lepton number of -1. Lepton number is conserved so that it is actually an antineutrino that is emitted together with electron emission β-decay and a neutrino together with positron emission.

The fact that the neutrino has (almost) zero mass is deduced by examining the end-point of the electron energy spectrum. For example for the decay

$$^3_1\mathrm{H} \rightarrow {}^3_2\mathrm{He} + e^- + \bar{\nu},$$

with a Q-value of 18.6 KeV,



For a massless neutrino its total (relativistic) energy can be arbitrarily small and the electron can carry energy up to the Q-value. If the neutrino has a mass, m_ν then the minimum energy that it can have is $m_\nu c^2$, and the electron energy spectrum drops off sharply at the end-point.

It is now known that neutrinos *do* have a tiny mass. The first hint of this was during the observation of the Supernova in 1987, when a burst of neutrinos were observed a few seconds after the burst of γ-rays, implying that the neutrinos had not travelled form the Supernova with exactly the speed of light. This was confirmed by neutrino observation experiments at the Kamiokande neutrino detector in Japan in 1999. However the mass of the neutrino is almost certainly smaller than 0.1 eV/c^2 (compared with the electron mass of 0.511 MeV/c^2). For our purposes we may neglect the neutrino mass.

8.2 Electron Capture

Nuclei which can β-decay emitting a positron and an neutrino, can also decay by another mechanism.

$$e^- + {}^{A}_{Z}\{P\} \rightarrow {}^{A}_{(Z-1)}\{D\} + \nu.$$

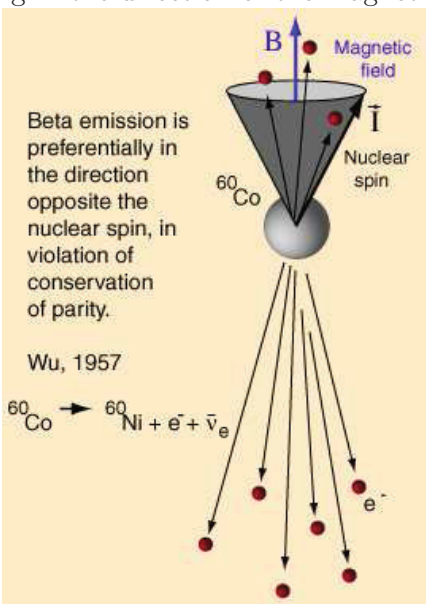
What happens here is that an atom can absorb an electron from one of the inner shells (usually the innermost shell, which is called the "K-shell") and be converted into an atom with one lower atomic number. The energy is entirely carried away by the neutrino and is nearly always undetected because neutrinos interact so weakly with matter.

8.3 Parity Violation

β-decay exhibits a further peculiarity. This was discovered in 1957 by C.S. Wu who observed the decay of radioactive cobalt into nickel

$$^{60}_{27}\mathrm{Co} \rightarrow {}^{60}_{28}\mathrm{Ni} + e^- + \bar{\nu}.$$

The cobalt sample was kept a low temperature and placed in a magnetic field so that the spin of the cobalt was pointing in the direction of the magnetic field.



She discovered that most of the electrons emerged in the opposite direction from the applied magnetic field. If we write $\mathbf{s}$ for the spin of the parent nucleus and $\mathbf{p_e}$ for the momentum of an emitted electron, this means that the average value of the scalar product $\mathbf{s} \cdot \mathbf{p_e}$ was negative. In order to balance the momentum the antineutrinos are usually emitted in the direction of the magnetic field, so that the average value of $\mathbf{s} \cdot \mathbf{p}_{\bar{\nu}}$ was positive.

Under the parity operation

$$\mathbf{r} \rightarrow -\mathbf{r}$$

and

$$\mathbf{p} \rightarrow -\mathbf{p}$$

but angular momentum which is defined as a vector product

$$\mathbf{L} = \mathbf{r} \times \mathbf{p},$$

is unchanged under parity

$$\mathbf{L} \rightarrow \mathbf{L}.$$

Spin is an internal angular momentum and so it also is unchanged under parity.

But this means that the scalar product $\mathbf{s} \cdot \mathbf{p_e}$ does change under parity

$$\mathbf{s} \cdot \mathbf{p_e} \rightarrow -\mathbf{s} \cdot \mathbf{p_e}$$

so that the fact that this quantity has a non-zero average value (or expectation value in quantum mechanics terms) means that the mechanism of β-decay violates parity conservation

If we viewed the above diagram in the corner of a mirrored room so that all the directions were reversed the spin would point in the same direction, but the electron direction would be reversed so that in that world the electrons would prefer to emerge in the direction of the magnetic field.

The spin of the daughter nucleus ${}^{60}_{28}\mathrm{Ni}$ is 4 (it is produced in an excited state) whereas that of the parent ${}^{60}_{27}\mathrm{Co}$ was 5, so that in order to compensate for unit of angular momentum lost (in the direction of the magnetic field) the angular momentum the antineutrinos and electrons have their spins in the direction of the magnetic field. This means that the antineutrinos have a spin component $+\frac{1}{2}$ in their direction of motion (in units of $\hbar$) whereas the electrons have a spin component $-\frac{1}{2}$ in their direction of motion. The sign of the component of the spin of a particle in its direction of motion is called the "helicity" of the particle. Neutrinos always have negative helicity (antineutrinos always have positive helicity). An electron can have component of spin either $+\frac{1}{2}$ or $-\frac{1}{2}$ in its direction of motion (either positive or negative helicity). However, the electrons emitted in β-decay usually have negative helicity (positrons emitted in β-decay usually have positive helicity). This means that the mechanism responsible for β-decay (called the "weak interaction") distinguish between positive and negative helicity and therefore violate parity.

7. Radioactive decay

Radioactive decay is the process in which an unstable nucleus spontaneously loses energy by emitting ionizing particles and radiation. This decay, or loss of energy, results in an atom of one type, called the **parent** nuclide, transforming to an atom of a different type, named the **daughter** nuclide.
The three principal modes of decay are called the alpha, beta and gamma decays. We already introduced the general principles of radioactive decay in Section 1.3 and we studied more in depth alpha decay in Section 3.3. In this chapter we consider the other two type of radioactive decay, beta and gamma decay, making use of our knowledge of quantum mechanics and nuclear structure.

7.1 Gamma decay

Gamma decay is the third type of radioactive decay. Unlike the two other types of decay, it does not involve a change in the element. It is just a simple decay from an excited to a lower (ground) state. In the process of course some energy is released that is carried away by a photon. Similar processes occur in atomic physics, however there the energy changes are usually much smaller, and photons that emerge are in the visible spectrum or x-rays.
The nuclear reaction describing gamma decay can be written as

$$ {}^{A}_{Z}X^{*} \rightarrow {}^{A}_{Z}X + \gamma $$

where * indicates an excited state.
We have said that the photon carries aways some energy. It also carries away momentum, angular momentum and parity (but no mass or charge) and all these quantities need to be conserved. We can thus write an equation for the energy and momentum carried away by the gamma-photon.
From special relativity we know that the energy of the photon (a massless particle) is

$$ E = \sqrt{m^2c^4 + p^2c^2} \quad \rightarrow \quad E = pc $$

(while for massive particles in the non-relativistic limit $v \ll c$ we have $E \approx mc^2 + \frac{p^2}{2m}$.) In quantum mechanics we have seen that the momentum of a wave (and a photon is well described by a wave) is $p = \hbar k$ with k the wave number. Then we have

$$ \boxed{E = \hbar k c = \hbar \omega_k} $$

This is the energy for photons which also defines the frequency $\omega_k = kc$ (compare this to the energy for massive particles, $E = \frac{\hbar^2 k^2}{2m}$).
Gamma photons are particularly energetic because they derive from nuclear transitions (that have much higher energies than e.g. atomic transitions involving electronic levels). The energies involved range from $E \sim .1 \div 10$MeV, giving $k \sim 10^{-1} \div 10^{-3}\text{fm}^{-1}$. Than the wavelengths are $\lambda = \frac{2\pi}{k} \sim 100 \div 10^{4}$fm, much longer than the typical nuclear dimensions.

Gamma ray spectroscopy is a basic tool of nuclear physics, for its ease of observation (since it's not absorbed in air), accurate energy determination and information on the spin and parity of the excited states.
Also, it is the most important radiation used in nuclear medicine.

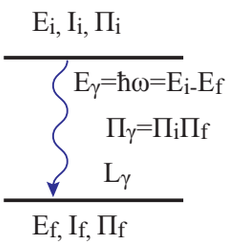


Fig. 42: Schematics of gamma decay

7.1.1 Classical theory of radiation

From the theory of electrodynamics it is known that an accelerating charge radiates. The power radiated is given by the integral of the energy flux (as given by the Poynting vector) over all solid angles. This gives the radiated power as:

$$\boxed{P = \frac{2}{3}\frac{e^2|a|^2}{c^3}}$$

where a is the acceleration. This is the so-called Larmor formula for a non-relativistic accelerated charge.
Example. As an important example we consider an electric dipole. An electric dipole can be considered as an oscillating charge, over a range r_0, such that the electric dipole is given by $d(t) = qr(t)$. Then the equation of motion is

$$r(t) = r_0\cos(\omega t)$$

and the acceleration

$$a = \ddot{r} = -r_0\omega^2\cos(\omega t)$$

Averaged over a period $T = 2\pi/\omega$, this is

$$\langle a^2\rangle = \frac{\omega}{2\pi}\int_0^T dt a(t) = \frac{1}{2}r_0^2\omega^4$$

Finally we obtain the radiative power for an electric dipole:

$$P_{E1} = \frac{1}{3}\frac{e^2\omega^4}{c^3}|\vec{r}_0|^2$$

A. Electromagnetic multipoles

In order to determine the classical e.m. radiation we need to evaluate the charge distribution that gives rise to it. The electrostatic potential of a charge distribution $\rho_e(r)$ is given by the integral:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_{Vol'}\frac{\rho_e(\vec{r'})}{|\vec{r}-\vec{r'}|}$$

When treating radiation we are only interested in the potential outside the charge and we can assume the charge (e.g. a particle!) to be well localized ($r' \ll r$). Then we can expand $\frac{1}{|\vec{r}-\vec{r'}|}$ in power series. First, we express explicitly the norm $|\vec{r}-\vec{r'}| = \sqrt{r^2+r'^2-2rr'\cos\vartheta} = r\sqrt{1+\left(\frac{r'}{r}\right)^2-2\frac{r'}{r}\cos\vartheta}$. We set $R = \frac{r'}{r}$ and $\epsilon = R^2 - 2R\cos\vartheta$: this is a small quantity, given the assumption $r' \ll r$. Then we can expand:

$$\frac{1}{|\vec{r}-\vec{r'}|} = \frac{1}{r}\frac{1}{\sqrt{1+\epsilon}} = \frac{1}{r}\left(1-\frac{1}{2}\epsilon+\frac{3}{8}\epsilon^2-\frac{5}{16}\epsilon^3+\dots\right)$$

Replacing ϵ with its expression we have:

$$\frac{1}{r}\frac{1}{\sqrt{1+\epsilon}} = \frac{1}{r}\left(1-\frac{1}{2}(R^2-2R\cos\vartheta)+\frac{3}{8}(R^2-2R\cos\vartheta)^2-\frac{5}{16}(R^2-2R\cos\vartheta)^3+\dots\right)$$

$$= \frac{1}{r}\left(1+[-\frac{1}{2}R^2+R\cos\vartheta]+[\frac{3}{8}R^4-\frac{3}{2}R^3\cos\vartheta+\frac{3}{2}R^2\cos^2\vartheta]+[-\frac{5R^6}{16}+\frac{15}{8}R^5\cos(\vartheta)-\frac{15}{4}R^4\cos^2(\vartheta)+\frac{5}{2}R^3\cos^3(\vartheta)]+\dots\right)$$

$$= \frac{1}{r}\left(1+R\cos\vartheta+R^2\left(\frac{3\cos^2\vartheta}{2}-\frac{1}{2}\right)+R^3\left(\frac{5\cos^3(\vartheta)}{2}-\frac{3\cos(\vartheta)}{2}\right)+\dots\right)$$

We recognized in the coefficients to the powers of R the Legendre Polynomials $P_l(\cos\vartheta)$ (with l the power of R^l, and note that for powers > 3 we should have included higher terms in the original ϵ expansion):

$$\frac{1}{r}\frac{1}{\sqrt{1+\epsilon}} = \frac{1}{r}\sum_{l=0}^{\infty} R^l P_l(\cos\vartheta) = \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l P_l(\cos\vartheta)$$

With this result we can as well calculate the potential:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r}\int_{Vol'}\rho(\vec{r'})\frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l P_l(\cos\vartheta)d\vec{r'}$$

The various terms in the expansion are the multipoles. The few lowest ones are :

$$\frac{1}{4\pi\epsilon_0}\frac{1}{r}\int_{Vol'}\rho(\vec{r'})\,d\vec{r'} = \frac{Q}{4\pi\epsilon_0 r} \qquad \text{Monopole}$$

$$\frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int_{Vol'}\rho(\vec{r'})r' P_1(\cos\vartheta)\,d\vec{r'} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int_{Vol'}\rho(\vec{r'})r'\cos\vartheta d\vec{r'} = \frac{\hat{r}\cdot\vec{d}}{4\pi\epsilon_0 r^2} \qquad \text{Dipole}$$

$$\frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\int_{Vol'}\rho(\vec{r'})r'^2 P_2(\cos\vartheta)\,d\vec{r'} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\int_{Vol'}\rho(\vec{r'})r'^2\left(\frac{3}{2}\cos^2\vartheta - \frac{1}{2}\right)d\vec{r'} \qquad \text{Quadrupole}$$

This type of expansion can be carried out as well for the magnetostatic potential and for the electromagnetic, time-dependent field.
At large distances, the lowest orders in this expansion are the only important ones. Thus, instead of considering the total radiation from a charge distribution, we can approximate it by considering the radiation arising from the first few multipoles: i.e. radiation from the electric dipole, the magnetic dipole, the electric quadrupole etc.
Each of these radiation terms have a peculiar angular dependence. This will be reflected in the quantum mechanical treatment by a specific angular momentum value of the radiation field associated with the multipole. In turns, this will give rise to selection rules determined by the addition rules of angular momentum of the particles and radiation involved in the radiative process.

7.1.2 Quantum mechanical theory

In quantum mechanics, gamma decay is expressed as a transition from an excited to a ground state of a nucleus. Then we can study the transition rate of such a decay via Fermi's Golden rule

$$W = \frac{2\pi}{\hbar}|\langle\psi_f|\,\hat{V}\,|\psi_i\rangle\,|^2\rho(E_f)$$

There are two important ingredients in this formula, the density of states $\rho(E_f)$ and the interaction potential $\hat{V}$.

A. Density of states

The density of states is defined as the number of available states per energy: $\rho(E_f) = \frac{d\,N_s}{d\,E_f}$, where N_s is the number of states. We have seen at various time the concept of degeneracy: as eigenvalues of an operator can be degenerate, there might be more than one eigenfunction sharing the same eigenvalues. In the case of the Hamiltonian, when there are degeneracies it means that more than one state share the same energy.
By considering the nucleus+radiation to be enclosed in a cavity of volume L^3, we have for the emitted photon a wavefunction represented by the solution of a particle in a 3D box that we saw in a Problem Set.
As for the 1D case, we have a quantization of the momentum (and hence of the wave-number k) in order to fit the wavefunction in the box. Here we just have a quantization in all 3 directions:

$$k_x = \frac{2\pi}{L}n_x, \quad k_y = \frac{2\pi}{L}n_y, \quad k_z = \frac{2\pi}{L}n_z,$$

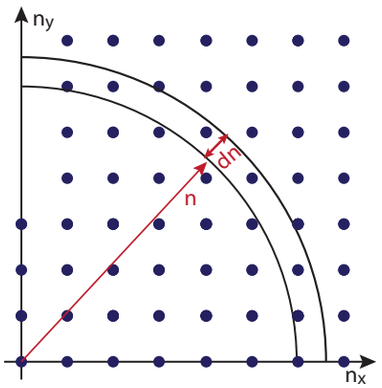


Fig. 43: Density of states: counting the states (2D)

(with n integers). Then, going to spherical coordinates, we can count the number of states in a spherical shell between n and $n+dn$ to be $dN_s = 4\pi n^2 dn$. Expressing this in terms of k, we have $dN_s = 4\pi k^2 dk \frac{L^3}{(2\pi)^3}$. If we consider just a small solid angle $d\Omega$ instead of 4π we have then the number of state $dN_s = \frac{L^3}{(2\pi)^3} k^2 dk d\Omega$. Since $E = \hbar kc = \hbar\omega$, we finally obtain the density of states:

$$\rho(E) = \frac{dN_s}{dE} = \frac{L^3}{(2\pi)^3} k^2 \frac{dk}{dE} d\Omega = \frac{L^3}{(2\pi)^3} \frac{k^2}{\hbar c} d\Omega = \frac{\omega^2}{\hbar c^3} \frac{L^3}{(2\pi)^3} d\Omega$$

B. The vector potential

Next we consider the potential causing the transition. The interaction of a particle with the e.m. field can be expressed in terms of the vector potential $\hat{\vec{A}}$ of the e.m. field as:

$$\hat{V} = \frac{e}{mc} \hat{\vec{A}} \cdot \hat{\vec{p}}$$

where $\hat{\vec{p}}$ is the particle's momentum. The vector potential $\hat{\vec{A}}$ in QM is an operator that can *create* or *annihilate* photons,

$$\hat{\vec{A}} = \sum_k \sqrt{\frac{2\pi\hbar c^2}{V\omega_k}} (\hat{a}_k e^{i\vec{k}\cdot\vec{r}} + \hat{a}_k^\dagger e^{-i\vec{k}\cdot\vec{r}}) \vec{\epsilon}_k$$

where $\hat{a}_k$ ($\hat{a}_k^\dagger$) annihilates (creates) one photon of momentum $\vec{k}$. Also, $\vec{\epsilon}_k$ is the polarization of the e.m. field. Since gamma decay (and many other atomic and nuclear processes) is able to create photons (or absorb them) it makes sense that the operator describing the e.m. field would be able to describe the creation and annihilation of photons. The second characteristic of this operator are the terms $\propto e^{-i\vec{k}\cdot\vec{r}}$ which describe a plane wave, as expected for e.m. waves, with momentum $\hbar k$ and frequency ck.

C. Dipole transition for gamma decay

To calculate the transition rate from the Fermi's Golden rule,

$$W = \frac{2\pi}{\hbar} |\langle \psi_f | \hat{V} | \psi_i \rangle|^2 \rho(E_f),$$

we are really only interested in the matrix element $\langle \psi_f | \hat{V} | \psi_i \rangle$, where the initial state does not have any photon, and the final has one photon of momentum $\hbar k$ and energy $\hbar\omega = \hbar kc$. Then, the only element in the sum above for the vector potential that gives a non-zero contribution will be the term $\propto \hat{a}_k^\dagger$, with the appropriate $\vec{k}$ momentum:

$$V_{if} = \frac{e}{mc} \sqrt{\frac{2\pi\hbar c^2}{V\omega_k}} \vec{\epsilon}_k \cdot \left\langle \hat{\vec{p}} e^{-i\vec{k}\cdot\vec{r}} \right\rangle$$

This can be simplified as follow. Remember that $[\hat{\vec{p}}^2, \hat{\vec{r}}] = -2i\hbar\hat{\vec{p}}$. Thus we can write, $\hat{\vec{p}} = \frac{i}{2\hbar}[\hat{\vec{p}}^2, \hat{\vec{r}}] = \frac{im}{\hbar}[\frac{\hat{\vec{p}}^2}{2m}, \hat{\vec{r}}] = \frac{im}{\hbar}[\frac{\hat{\vec{p}}^2}{2m} + V_{nuc}(\hat{\vec{r}}), \hat{\vec{r}}]$. We introduced the nuclear Hamiltonian $\mathcal{H}_{nuc} = \frac{\hat{\vec{p}}^2}{2m} + V_{nuc}(\hat{\vec{r}})$: thus we have $\hat{\vec{p}} = \frac{im}{\hbar}[\mathcal{H}_{nuc}, \hat{\vec{r}}]$. Taking the expectation value

$$\langle \psi_f | \hat{\vec{p}} | \psi_i \rangle = \frac{im}{\hbar} \left(\langle \psi_f | \mathcal{H}_{nuc} \hat{\vec{r}} | \psi_i \rangle - \langle \psi_f | \hat{\vec{r}} \mathcal{H}_{nuc} | \psi_i \rangle \right)$$

and remembering that $|\psi_{i,f}\rangle$ are eigenstates of the Hamiltonian, we have

$$\langle \psi_f | \hat{\vec{p}} | \psi_i \rangle = \frac{im}{\hbar} (E_f - E_i) \langle \psi_f | \hat{\vec{r}} | \psi_i \rangle = im\omega_k \langle \psi_f | \hat{\vec{r}} | \psi_i \rangle,$$

where we used the fact that $(E_f - E_i) = \hbar\omega_k$ by conservation of energy. Thus we obtain

$$V_{if} = \frac{e}{mc} \sqrt{\frac{2\pi\hbar c^2}{V\omega_k}} im\omega \vec{\epsilon}_k \cdot \left\langle \hat{\vec{r}} e^{-i\vec{k}\cdot\vec{r}} \right\rangle = i \sqrt{\frac{2\pi\hbar e^2 \omega_k}{V}} \vec{\epsilon}_k \cdot \left\langle \hat{\vec{r}} e^{-i\vec{k}\cdot\vec{r}} \right\rangle$$

We have seen that the wavelengths of gamma photons are much larger than the nuclear size. Then $\vec{k}\cdot\vec{r}\ll 1$ and we can make an expansion in series : $e^{-\vec{k}\cdot\vec{r}}\sim\sum_l \frac{1}{l!}(-i\vec{k}\cdot\vec{r})^l=\sum_l \frac{1}{l!}(-ikr\cos\vartheta)^l$. This series is very similar in meaning to the multipole series we saw for the classical case.
For example, for $l=0$ we obtain:

$$V_{if}=\sqrt{\frac{2\pi\hbar e^2\omega_k}{V}}\left\langle\hat{\vec{r}}\right\rangle\cdot\vec{\epsilon}_k$$

which is the dipolar approximation, since it can be written also using the electric dipole operator $e\hat{\vec{r}}$.
The angle between the polarization of the e.m. field and the position $\hat{\vec{r}}$ is $\left\langle\hat{\vec{r}}\right\rangle\cdot\vec{\epsilon}=\left\langle\hat{\vec{r}}\right\rangle\sin\vartheta$
The transition rate for the dipole radiation, $W\equiv\lambda(E1)$ is then:

$$\lambda(E1)=\frac{2\pi}{\hbar}|\langle\psi_f|\,\hat{V}\,|\psi_i\rangle\,|^2\rho(E_f)=\frac{\omega^3}{2\pi c^3\hbar}|\left\langle\hat{\vec{r}}\right\rangle|^2\sin^2\vartheta\,d\Omega$$

and integrating over all possible direction of emission ($\int_0^{2\pi}d\varphi\int_0^{\pi}(\sin^2\vartheta)\sin\vartheta d\vartheta=2\pi\frac{4}{3}$):

$$\lambda(E1)=\frac{4}{3}\frac{e^2\omega^3}{\hbar c^3}|\left\langle\hat{\vec{r}}\right\rangle|^2$$

Multiplying the transition rate (or photons emitted per unit time) by the energy of the photons emitted we obtain the radiated power, $P=W\hbar\omega$:

$$P=\frac{4}{3}\frac{e^2\omega^4}{c^3}|\left\langle\hat{\vec{r}}\right\rangle|^2$$

Notice the similarity of this formula with the classical case:

$$P_{E1}=\frac{1}{3}\frac{e^2\omega^4}{c^3}|\vec{r}_0|^2$$

We can estimate the transition rate by using a typical energy $E=\hbar\omega$ for the photon emitted (equal to a typical energy difference between excited and ground state nuclear levels) and the expectation value for the dipole ($|\left\langle\hat{\vec{r}}\right\rangle|\sim R_{nuc}\approx r_0A^{1/3}$). Then, the transition rate is evaluated to be

$$\lambda(E1)=\frac{e^2}{\hbar c}\frac{E^3}{(\hbar c)^3}r_0^2A^{2/3}=1.0\times10^{14}A^{2/3}E^3$$

(with E in MeV). For example, for $A=64$ and $E=1$MeV the rate is $\lambda\approx1.6\times10^{15}s^{-1}$ or $\tau=10^{-15}$ (femtoseconds!) for $E=0.1$MeV τ is on the order of picoseconds.
Obs. Because of the large energies involved, very fast processes are expected in the nuclear decay from excited states, in accordance with Fermi's Golden rule and the energy/time uncertainty relation.

7.1.3 Extension to Multipoles

We obtained above the transition rate for the electric dipole, i.e. when the interaction between the nucleus and the e.m. field is described by an electric dipole and the emitted radiation has the character of electric dipole radiation. This type of radiation can only carry out of the nucleus one quantum of angular momentum (i.e. $\Delta l=\pm1$, between excited and ground state). In general, excited levels differ by more than 1 l, thus the radiation emitted need to be a higher multipole radiation in order to conserve angular momentum.

A. Electric Multipoles

We can go back to the expansion of the radiation interaction in multipoles:

$$\hat{V}\sim\sum_l\frac{1}{l!}(i\hat{\vec{k}}\cdot\hat{\vec{r}})^l$$

Then the transition rate becomes:

$$\lambda(El)=\frac{8\pi(l+1)}{l[(2l+1)!!]^2}\frac{e^2}{\hbar c}\left(\frac{E}{\hbar c}\right)^{2l+1}\left(\frac{3}{l+3}\right)^2c\left\langle|\hat{\vec{r}}|\right\rangle^{2l}$$

Notice the strong dependence on the l quantum number. Setting again $|\left\langle\hat{\vec{r}}\right\rangle|\sim r_0A^{1/3}$ we also have a strong dependence on the mass number.
Thus, we have the following estimates for the rates of different electric multipoles:

- $\lambda(E1) = 1.0 \times 10^{14} A^{2/3} E^3$
- $\lambda(E2) = 7.3 \times 10^{7} A^{4/3} E^5$
- $\lambda(E3) = 34 A^2 E^7$
- $\lambda(E4) = 1.1 \times 10^{-5} A^{8/3} E^9$

B. Magnetic Multipoles

The e.m. potential can also contain magnetic interactions, leading to magnetic transitions. The transition rates can be calculated from a similar formula:

$$\lambda(Ml) = \frac{8\pi(l+1)}{l[(2l+1)!!]^2} \frac{e^2}{\hbar c} \frac{E}{\hbar c}^{2l+1} \left(\frac{3}{l+3}\right)^2 c \left\langle |\hat{\vec{r}}| \right\rangle^{2l-2} \left[\frac{\hbar}{m_p c}\left(\mu_p - \frac{1}{l+1}\right)\right]$$

where μ_p is the magnetic moment of the proton (and m_p its mass).
Estimates for the transition rates can be found by setting $\mu_p - \frac{1}{l+1} \approx 10$:

- $\lambda(M1) = 5.6 \times 10^{13} E^3$
- $\lambda(M2) = 3.5 \times 10^{7} A^{2/3} E^5$
- $\lambda(M3) = 16 A^{4/3} E^7$
- $\lambda(M4) = 4.5 \times 10^{-6} A^2 E^9$

7.1.4 Selection Rules

The angular momentum must be conserved during the decay. Thus the difference in angular momentum between the initial (excited) state and the final state is carried away by the photon emitted. Another conserved quantity is the total parity of the system.

A. Parity change

The parity of the gamma photon is determined by its character, either magnetic or electric multipole. We have

$$\Pi_\gamma(El) = (-1)^l \quad \text{Electric multipole}$$

$$\Pi_\gamma(Ml) = (-1)^{l-1} \quad \text{Magnetic multipole}$$

Then if we have a parity change from the initial to the final state $\Pi_i \to \Pi_f$ this is accounted for by the emitted photon as:

$$\Pi_\gamma = \Pi_i \Pi_f$$

This of course limits the type of multipole transitions that are allowed given an initial and final state.

$$\Delta\Pi = \text{no} \to \text{Even Electric, Odd Magnetic}$$

$$\Delta\Pi = \text{yes} \to \text{Odd Electric, Even Magnetic}$$

B. Angular momentum

From the conservation of the angular momentum:

$$\hat{\vec{I}}_i = \hat{\vec{I}}_f + \hat{\vec{L}}_\gamma$$

the allowed values for the angular momentum quantum number of the photon, l, are restricted to

$$l_\gamma = |I_i - I_f|, \ldots, I_i + I_f$$

Once the allowed l have been found from the above relationship, the character (magnetic or electric) of the multipole is found by looking at the parity.
In general then, the most important transition will be the one with the lowest allowed l, Π. Higher multipoles are also possible, but they are going to lead to much slower processes.

Multipolarity	Angular Momentum l	Parity Π	Multipolarity	Angular Momentum l	Parity Π
M1	1	+	E1	1	-
M2	2	-	E2	2	+
M3	3	+	E3	3	-
M4	4	-	E4	4	+
M5	5	+	E5	5	-

Table 3: Angular momentum and parity of the gamma multipoles

C. Dominant Decay Modes

In general we have the following predictions of which transitions will happen:

1. The lowest permitted multipole dominates
2. Electric multipoles are more probable than the same magnetic multipole by a factor $\sim 10^2$ (however, which one is going to happen depends on the parity)
$$\frac{\lambda(El)}{\lambda(Ml)} \approx 10^2$$
3. Emission from the multipole $l+1$ is 10^{-5} times less probable than the l-multipole emission.
$$\frac{\lambda(E,l+1)}{\lambda(El)} \approx 10^{-5}, \qquad \frac{\lambda(M,l+1)}{\lambda(Ml)} \approx 10^{-5}$$
4. Combining 2 and 3, we have:
$$\frac{\lambda(E,l+1)}{\lambda(Ml)} \approx 10^{-3}, \qquad \frac{\lambda(M,l+1)}{\lambda(El)} \approx 10^{-7}$$
Thus $E2$ competes with $M1$ while that's not the case for $M2$ vs. $E1$

D. Internal conversion

What happen if no allowed transitions can be found? This is the case for even-even nuclides, where the decay from the 0^+ excited state must happen without a change in angular momentum. However, the photon always carries some angular momentum, thus gamma emission is impossible.
Then another process happens, called **internal conversion**:

$$^A_Z X^* \rightarrow ^A_Z X + e^-$$

where $^A_Z X$ is a ionized state and e^- is one of the atomic electrons.
Besides the case of even-even nuclei, internal conversion is in general a competing process of gamma decay (see Krane for more details).

Chapter 4

The Liquid Drop Model

4.1 Some Nuclear Nomenclature

- **Nucleon:** A proton or neutron.
- **Atomic Number, Z:** The number of protons in a nucleus.
- **Atomic Mass number, A:** The number of nucleons in a nucleus.
- **Nuclide:** A nucleus with a specified value of A and Z. This is usually written as ${}^{A}_{Z}\{Ch\}$ where Ch is the Chemical symbol. e.g. ${}^{56}_{28}$Ni means Nickel with 28 protons and a further 28 neutrons.
- **Isotope:** Nucleus with a given atomic number but different atomic mass number, i.e. different number of neutrons. Isotopes have very similar atomic and chemical behaviour but may have very different nuclear properties.
- **Isotone:** Nulceus with a given number of neutrons but a different number of protons (fixed (A-Z)).
- **Isobar:** Nucleus with a given A but a different Z.
- **Mirror Nuclei:** Two nuclei with odd A in which the number of protons in one nucleus is equal to the number of neutrons in the other and vice versa.

4.2 Binding Energy

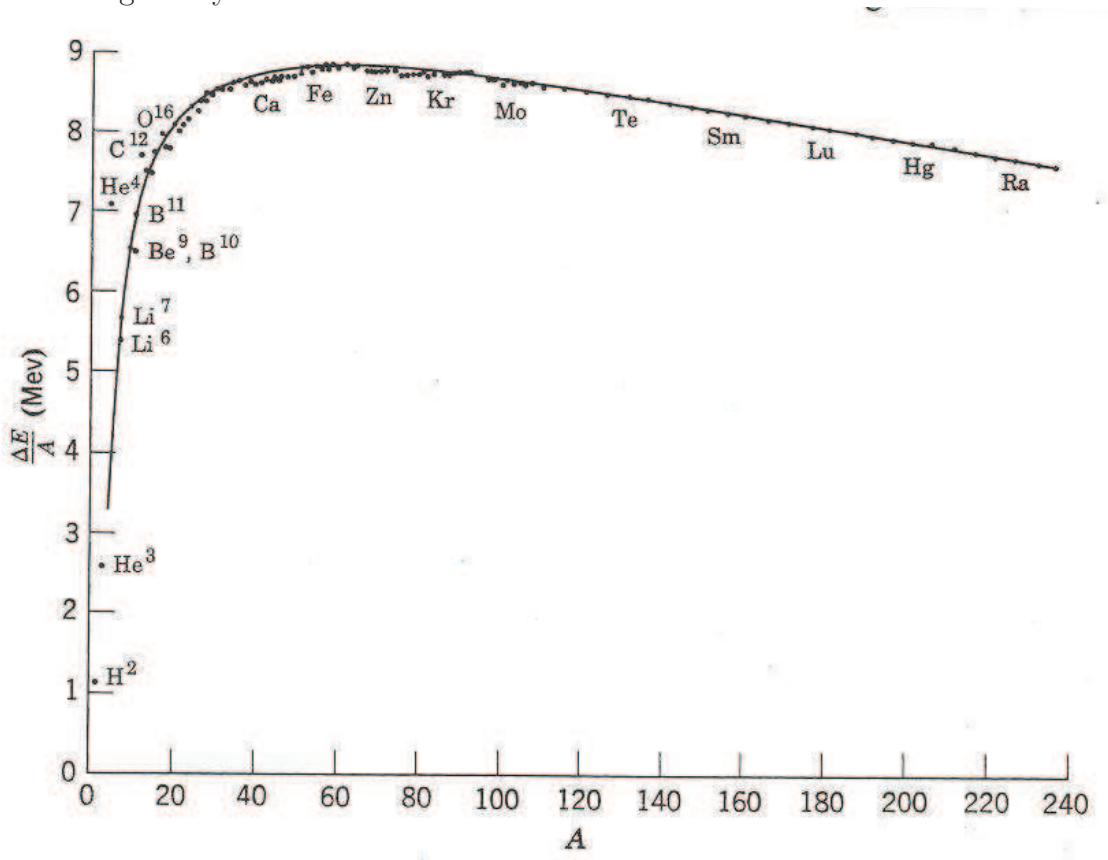
The mass of a nuclide is given by

$$m_N \;=\; Z\,m_p \;+\; (A-Z)\,m_n \;-\; B(A,Z)/c^2,$$

where $B(A,Z)$ is the binding energy of the nucleons and depends on both Z and A. The binding energy is due to the strong short-range nuclear forces that bind the nucleons together.

Unlike Coulomb binding these cannot even in principle be calculated analytically as the strong forces are much less well understood than electromagnetism.

Binding energies per nucleon increase sharply as A increases, peaking at iron (Fe) and then decreasing slowly for the more massive nuclei.



The binding energy divided by c^2 is sometimes known as the "mass defect".

4.3 Semi-Empirical Mass Formula

For most nuclei (nuclides) with $A > 20$ the binding energy is well reproduced by a semi-empirical formula based on the idea the the nucleus can be thought of as a liquid drop.

1. **Volume term:** Each nucleon has a binding energy which binds it to the nucleus. Therefore we get a term proportional to the volume i.e. proportional to A.

$$a_V\, A$$

This term reflects the short-range nature of the strong forces. If a nucleon interacted with *all* other nucleons we would expect an energy term of proportional to $A(A-1)$, but the fact that it turns out to be proportional to A indicates that a nucleon only interact with its nearest neighbours.

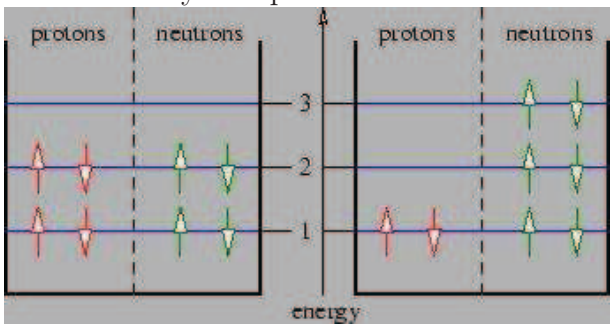
2. **Surface term:** The nucleons at the surface of the 'liquid drop' only interact with other nucleons inside the nucleus, so that their binding energy is reduced. This leads to a reduction of the binding energy proportional to the surface area of the drop, i.e. proportional to $A^{2/3}$

$$-a_S\, A^{2/3}.$$

3. **Coulomb term:** Although the binding energy is mainly due to the strong nuclear force, the binding energy is reduced owing to the Coulomb repulsion between the protons. We expect this to be proportional to the square of the nuclear charge, Z, (the electromagnetic force is long-range so each proton interact with all the others), and by Coulomb's law it is expected to be inversely proportional to the nuclear radius, (the Coulomb energy of a charged sphere of radius R and charge Q is $3Q^2/(20\pi\epsilon_0 R)$) The Coulomb term is therefore proportional to $1/A^{1/3}$

$$-a_C \frac{Z^2}{A^{1/3}}$$

4. **Asymmetry term:** This is a quantum effect arising from the Pauli exclusion principle which only allows two protons or two neutrons (with opposite spin direction) in each energy state. If a nucleus contains the same number of protons and neutrons then for each type the protons and neutrons fill to the same maximum energy level (the 'fermi level'). If, on the other hand, we exchange one of the neutrons by a proton then that proton would be required by the exclusion principle to occupy a higher energy state, since all the ones below it are already occupied.



The upshot of this is that nuclides with $Z = N = (A-Z)$ have a higher binding energy, whereas for nuclei with different numbers of protons and neutrons (for fixed A) the binding energy decreases as the square of the number difference. The spacing between energy levels is inversely proportional to the volume of the nucleus - this can be seen by treating the nucleus as a three-dimensional potential well- and therefore inversely proportional to A. Thus we get a term

$$-a_A \frac{(Z-N)^2}{A}$$

5. **Pairing term:** It is found experimentally that two protons or two neutrons bind more strongly than one proton and one neutron.
In order to account for this experimentally observed phenomenon we add a term to the binding energy if number of protons and number of neutrons are *both* even, we subtract

the same term if these are *both* odd, and do nothing if one is odd and the other is even. Bohr and Mottelson showed that this term was inversely proportional to the square root of the atomic mass number.
We therefore have a term

$$\frac{\left((-1)^Z + (-1)^N\right)}{2} \frac{a_P}{A^{1/2}}.$$

The complete formula is, therefore

$$B(A,Z) \;=\; a_V\, A \,-\, a_S\, A^{2/3} \,-\, a_C \frac{Z^2}{A^{1/3}} \,-\, a_A \frac{(Z-N)^2}{A} \,+\, \frac{\left((-1)^Z + (-1)^N\right)}{2} \frac{a_P}{A^{1/2}}$$

From fitting to the measured nuclear binding energies, the values of the parameters a_V, a_S, a_C, a_A, a_P are

$$\begin{aligned} a_V &= 15.56 \text{ MeV} \\ a_S &= 17.23 \text{ MeV} \\ a_C &= 0.697 \text{ MeV} \\ a_A &= 23.285 \text{ MeV} \\ a_P &= 12.0 \text{ MeV} \end{aligned}$$

For most nuclei with $A > 20$ this simple formula does a very good job of determining the binding energies - usually better than 0.5%.

For example we estimate the binding energy per nucleon of $^{80}_{35}$Br (Bromine), for which Z=35, A=80 ($N = 80 - 35 = 45$) and insert into the above formulae to get

$$\begin{aligned} \text{Volume term:} \quad & (15.56 \times 80) \;=\; 1244.8\,\text{MeV} \\ \text{Surface term:} \quad & (-17.23 \times (80)^{2/3}) \;=\; -319.9\,\text{MeV} \\ \text{Coulomb term:} \quad & \left(\frac{0.697 \times 35^2}{(80)^{1/3}}\right) \;=\; -198.2\,\text{MeV} \\ \text{Asymmetry term:} \quad & \left(\frac{23.285 \times (45-35)^2}{80}\right) \;=\; -29.1\,\text{MeV} \\ \text{Pairing term:} \quad & \left(\frac{-12.0}{(80)^{1/2}}\right) \;=\; -1.3\,\text{MeV} \end{aligned}$$

Note that we *subtract* the pairing term since both (A-Z) and Z are odd. This gives a total binding energy of 696.3 MeV. The measured value is 694.2 MeV.

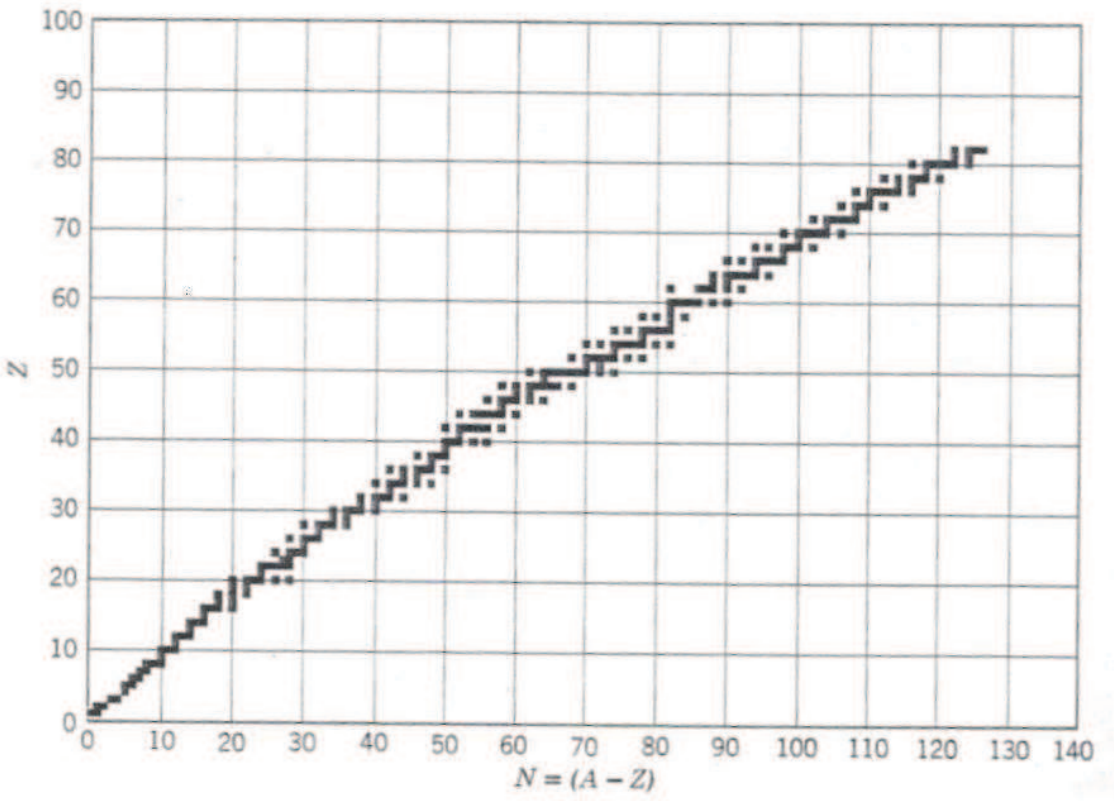
In order to calculate the mass of the nucleus we *subtract* this binding energy (divided by c^2) from the total mass of the protons and neutrons ($m_p = 938.4 MeV/c^2$, $m_n = 939.6 MeV/c^2$)

$$m_{Br} \;=\; 35 m_p + 45 m_n - 696.1 MeV/c^2 \;=\; 74417 \text{ Mev/c}^2.$$

Nuclear masses are nowadays usually quoted in MeV/c^2 but are still sometimes quoted in atomic mass units, defined to be 1/12 of the *atomic* mass of $^{12}_{6}$C (Carbon). The conversion factor is

$$1\,a.u. \;=\; 931.5\ \mathrm{MeV/c^2}$$

Since different isotopes have different atomic mass numbers they will have different binding energies and some isotopes will be more stable than others. It turns out (and can be seen by looking for the most stable isotopes using the semi-empirical mass formula) that for the lighter nuclei the stable isotopes have approximately the same number of neutrons as protons, but above A $\sim$ 20 the number of neutrons required for stability increases up to about one and a half times the number of protons for the heaviest nuclei.



Qualitatively, the reason for this arises from the Coulomb term. Protons bind less tightly than neutrons because they have to overcome the Coulomb repulsion between them. It is therefore energetically favourable to have more neutrons than protons. Up to a certain limit this Coulomb effect beats the asymmetry effect which favours equal numbers of protons and neutrons.

Chapter 5

Nuclear Shell Model

5.1 Magic Numbers

The binding energies predicted by the Liquid Drop Model underestimate the actual binding energies of "magic nuclei" for which either the number of neutrons $N = (A - Z)$ or the number of protons, Z is equal to one of the following "magic numbers"

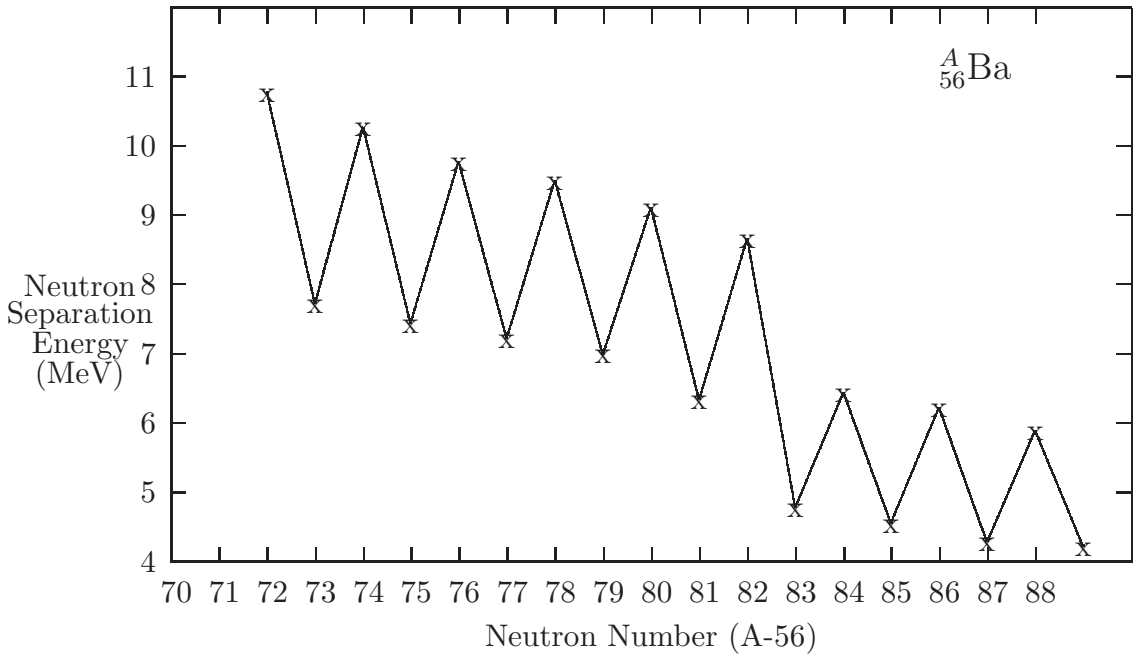
$$2,\ 8,\ 20,\ 28,\ 50,\ 82,\ 126.$$

This is particularly the case for "doubly magic" nuclei in which *both* the number of neutrons and the number of protons are equal to magic numbers.

For example for $^{56}_{28}$Ni (nickel) the Liquid Drop Model predicts a binding energy of 477.7 MeV, whereas the measured value is 484.0 MeV. Likewise for $^{132}_{50}$Sn (tin) the Liquid Drop model predicts a binding energy of 1084 MeV, whereas the measured value is 1110 MeV.

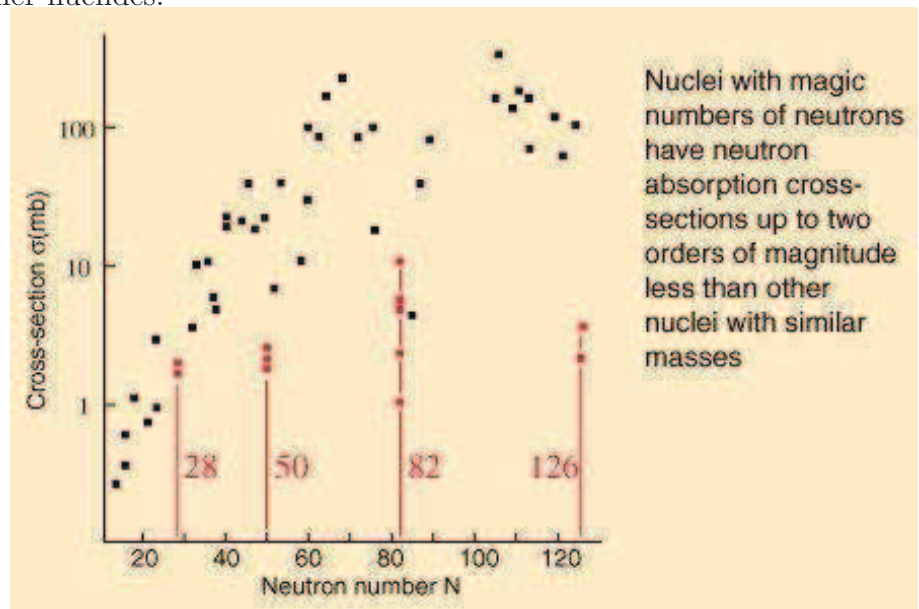
There are other special features of magic nuclei:

- The neutron (proton) separation energies (the energy required to remove the last neutron (proton)) peaks if N (Z) is equal to a magic number.

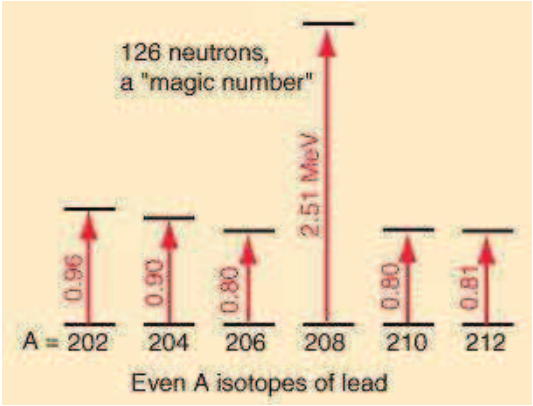


- There are more stable isotopes if Z is a magic number, and more stable isotones if N is a magic number.

- If N is magic number then the cross-section for neutron absorption is much lower than for other nuclides.



- The energies of the excited states are much higher than the ground state if either N or Z or both are magic numbers.



- Elements with Z equal to a magic number have a larger natural abundance than those of nearby elements.

5.2 Shell Model

These magic numbers can be explained in terms of the Shell Model of the nucleus, which considers each nucleon to be moving in some potential and classifies the energy levels in terms of quantum numbers $n\,l\,j$, in the same way as the wavefunctions of individual electrons are classified in Atomic Physics.

For a spherically symmetric potential the wavefunction (neglecting its spin for the moment) for any nucleon whose coordinates from the centre of the nucleus are given by polar coordinates $(r,\,\theta,\,\phi)$ is of the form

$$\Psi_{nlm} \;=\; R_{nl}(r)Y_l^m(\theta,\phi).$$

The energy eigenvalues will depend on the principle quantum number, n, and the orbital angular momentum, l, but are degenerate in the magnetic quantum number m. These energy levels come in 'bunches' called "shells" with a large energy gap just above each shell.

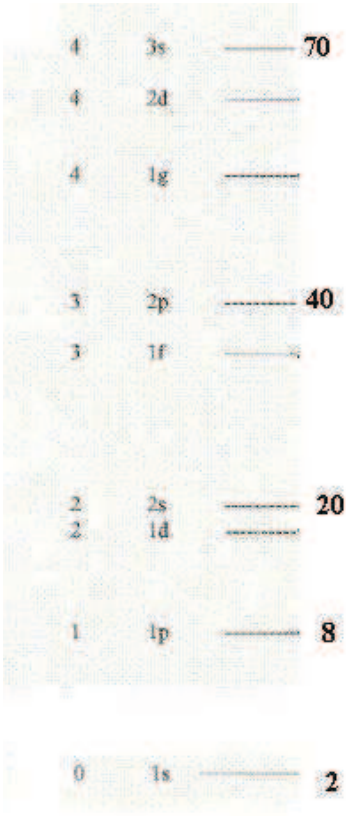
In their ground state the nucleons fill up the available energy levels from the bottom upwards with two protons (neutrons) in each available proton (neutron) energy level.

Unlike Atomic Physics we do not even understand in principle what the properties of this potential are - so we need to take a guess.

A simple harmonic potential (i.e. $V(r) \propto r^2$) would yield equally spaced energy levels and we would not see the shell structure and hence the magic numbers.

It turns out that once again the Saxon-Woods model is a reasonable guess, i.e.

$$V(r) \;=\; -\frac{V_0}{1+\exp\left(((r-R)/\delta)\right)}$$



For such a potential it turns out that the lowest level is $1s$ (i.e. $n = 1,\ l = 0$) which can contain up to 2 protons or neutrons. Then comes $1p$ which can contain up to a further 6 protons (neutrons). This explains the first 2 magic numbers (2 and 8). Then there is the level $1d$, but this is quite close in energy to $2s$ so that they form the same shell. This allows a further 2+10 protons (neutrons) giving us the next magic number of 20.

The next two levels are $1f$ and $2p$ which are also quite close together and allow a further 6+14 protons (neutrons). This would suggest that the next magic number was 40 - but experimentally it is known to be 50.

The solution to this puzzle lies in the spin-orbit coupling. Spin-orbit coupling - the interaction between the orbital angular momentum and spin angular momentum occurs in Atomic Physics. In Atomic Physics, the origin is magnetic and the effect is a small correction. In the case of nuclear binding the effect is about 20 times larger, and it comes from a term in the nuclear potential itself which is proportional to $\mathbf{L} \cdot \mathbf{S}$, i.e.

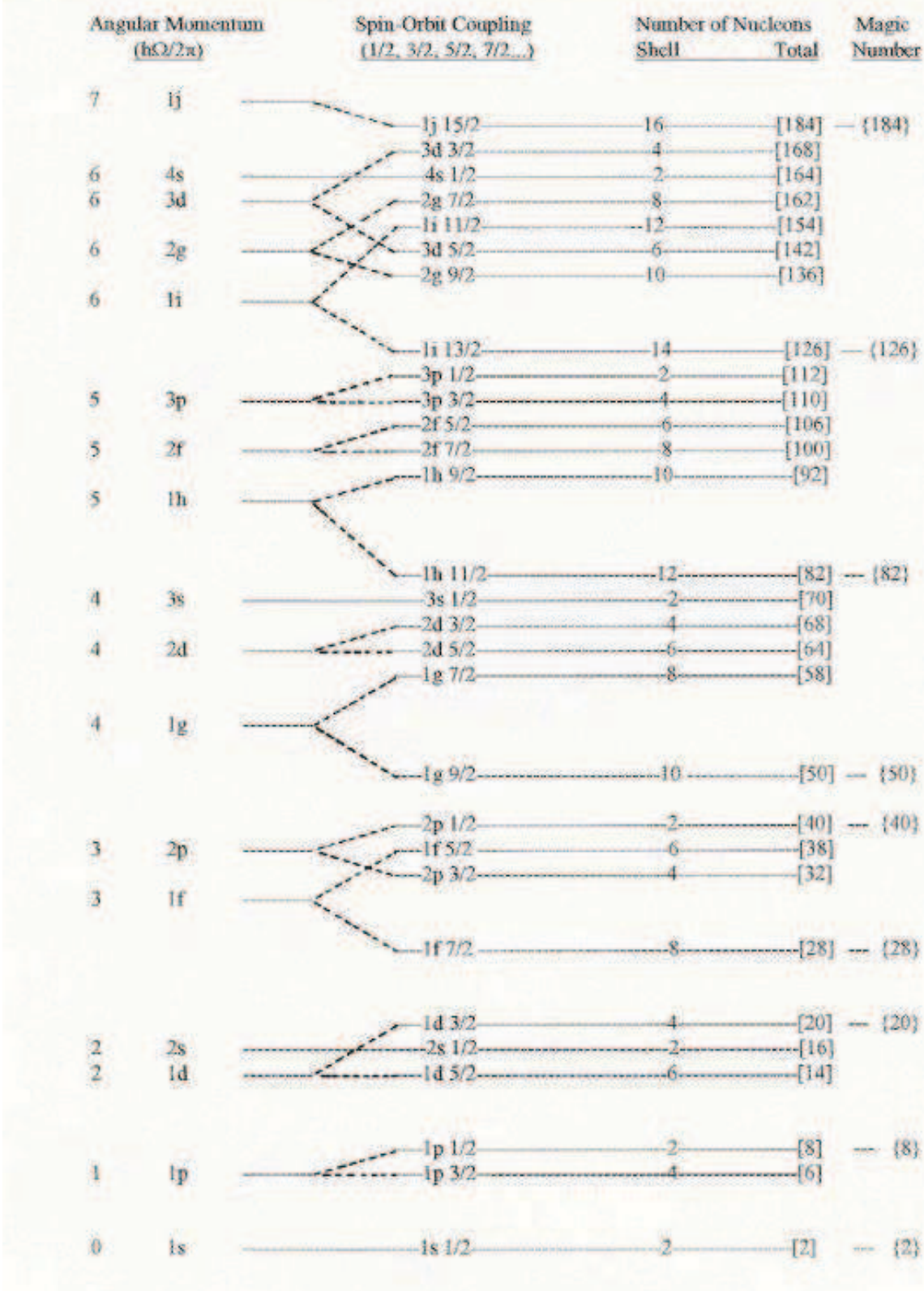
$$V(r) \ \rightarrow \ V(r) \ + \ W(r)\mathbf{L} \cdot \mathbf{S}.$$

As in the case of Atomic Physics (j-j coupling scheme) the orbital and spin angular momenta of the nucleons combine to give a total angular momentum j which can take the values $j = l + \frac{1}{2}$ or $j = l - \frac{1}{2}$. The spin-orbit coupling term leads to an energy shift proportional to

$$j(j+1) - l(l+1) - s(s+1), \quad (s = 1/2).$$

A further feature of this spin-orbit coupling in nuclei is that the energy split is in the opposite sense from its effect in Atomic Physics, namely that states with higher j have *lower energy*.

Nuclear Shell Structure



We see that this large spin-orbit effect leads to crossing over of energy levels into different shells. For example the state above the $2p$ state is $1g$ (l=4), which splits into $1g_{\frac{9}{2}}$, $(j = \frac{9}{2})$ and $1g_{\frac{7}{2}}$, $(j = \frac{7}{2})$. The energy of the $1g_{\frac{9}{2}}$ state is sufficiently low that it joins the shell below, so that this fourth shell now consists of $1f_{\frac{7}{2}}, 2p_{\frac{3}{2}}$, $1f_{\frac{5}{2}}$, $2p_{\frac{1}{2}}$ and $1g_{\frac{9}{2}}$. The maximum occupancy of this state ($(2j+1)$ protons (neutrons) for each j) is now 8+4+6+2+10=30, which added to the previous magic number, 20, gives the next observed magic number of 50.

Further up, it is the $1h$ state that undergoes a large splitting into $1h_{\frac{11}{2}}$ and $1h_{\frac{9}{2}}$, with the $1h_{\frac{11}{2}}$ state joining the lower shell.

5.3 Spin and Parity of Nuclear Ground States.

Nuclear states have an intrinsic spin and a well defined parity, $\eta = \pm 1$, defined by the behaviour of the wavefunction for all the nucleons under reversal of their coordinates with the centre of the nucleus at the origin.

$$\Psi(-\mathbf{r_1}, -\mathbf{r_2} \cdots - \mathbf{r_A}) \;=\; \eta\Psi(\mathbf{r_1}, \mathbf{r_2} \cdots \mathbf{r_A})$$

.

The spin and parity of nuclear ground states can usually be determined from the shell model. Protons and neutrons tend to pair up so that the spin of each pair is zero and each pair has even parity ($\eta = 1$). Thus we have

- Even-even nuclides (both Z and A even) have zero intrinsic spin and even parity.
- Odd A nuclei have one unpaired nucleon. The spin of the nucleus is equal to the j-value of that unpaired nucleon and the parity is $(-1)^l$, where l is the orbital angular momentum of the unpaired nucleon.

  For example $^{47}_{22}$Ti (titanium) has an even number of protons and 25 neutrons. 20 of the neutrons fill the shells up to magic number 20 and there are 5 in the $1f_{\frac{7}{2}}$ state $(l = 3,\ j = \frac{7}{2})$ Four of these form pairs and the remaining one leads to a nuclear spin of $\frac{7}{2}$ and parity $(-1)^3 = -1$.
- Odd-odd nuclei. In this case there is an unpaired proton whose total angular momentum is j_1 and an unpaired neutron whose total angular momentum is j_2. The total spin of the nucleus is the (vector) sum of these angular momenta and can take values between $|j_1 - j_2|$ and $|j_1 + j_2|$ (in unit steps). The parity is given by $(-1)^{(l_1+l_2)}$, where l_1 and l_2 are the orbital angular momenta of the unpaired proton and neutron respectively.

  For example $^{6}_{3}$Li (lithium) has 3 neutrons and 3 protons. The first two of each fill the $1s$ level and the thrid is in the $1p_{\frac{3}{2}}$ level. The orbital angular mometum of each is $l = 1$ so the parity is $(-1) \times (-1) = +1$ (even), but the spin can be anywhere between 0 and 3.

5.4 Magnetic Dipole Moments

Since nuclei with an odd number of protons and/or neutrons have intrinsic spin they also in general possess a magnetic dipole moment.

The unit of magnetic dipole moment for a nucleus is the "nuclear magneton" defined as

$$\mu_N \;=\; \frac{e\hbar}{2m_p},$$

which is analogous to the Bohr magneton but with the electron mass replaced by the proton mass. It is defined such that the magnetic moment due to a proton with orbital angular momentum $\mathbf{l}$ is $\mu_N\,\mathbf{l}$.

Experimentally it is found that the magnetic moment of the proton (due to its spin) is

$$\mu_p \;=\; 2.79\mu_N \;=\; 5.58\mu_N s, \quad \left(s=\frac{1}{2}\right)$$

and that of the neutron is

$$\mu_n \;=\; -1.91\mu_N \;=\; -3.82\mu_N s, \quad \left(s=\frac{1}{2}\right)$$

If we apply a magnetic field in the z-direction to a nucleus then the unpaired proton with orbital angular momentum $\mathbf{l}$, spin $\mathbf{s}$ and total angular momentum $\mathbf{j}$ will give a contribution to the $z-$ component of the magnetic moment

$$\mu^z \;=\; (5.58s^z \;+\; l^z)\,\mu_N.$$

As in the case of the Zeeman effect, the vector model may be used to express this as

$$\mu^z \;=\; \frac{(5.58 < \mathbf{s}\cdot\mathbf{j} > + < \mathbf{l}\cdot\mathbf{j} >)}{< \mathbf{j}^2 >} j^z\,\mu_N$$

using

$$\begin{aligned}
< \mathbf{j}^2 > &= j(j+1)\hbar^2 \\
< \mathbf{s}\cdot\mathbf{j} > &= \frac{1}{2}\left(< \mathbf{j}^2 > + < \mathbf{s}^2 > - < \mathbf{l}^2 >\right) \\
&= \frac{\hbar^2}{2}\left(j(j+1) \;+\; s(s+1) \;-\; l(l+1)\right) \\
< \mathbf{l}\cdot\mathbf{j} > &= \frac{1}{2}\left(< \mathbf{j}^2 > + < \mathbf{l}^2 > - < \mathbf{s}^2 >\right) \\
&= \frac{\hbar^2}{2}\left(j(j+1) \;+\; l(l+1) \;-\; s(s+1)\right) \qquad (5.4.1)
\end{aligned}$$

We end up with expression for the contribution to the magnetic moment

$$\mu \;=\; \frac{5.58\left(j(j+1) \;+\; s(s+1) \;-\; l(l+1)\right) + \left(j(j+1) \;+\; l(l+1) \;-\; s(s+1)\right)}{2j(j+1)}\, j\,\mu_N$$

and for a neutron with orbital angular momentum l' and total angular momentum j' we get (not contribution from the orbital angular momentum because the neutron is uncharged)

$$\mu \;=\; -\frac{3.82\,(j'(j'+1)\;+\;s(s+1)\;-\;l'(l'+1))}{2j'(j'+1)}j'\,\mu_N$$

Thus, for example if we consider the nuclide $^7_3$Li for which there is an unpaired proton in the $2p_{\frac{3}{2}}$ state ($l=1,\ j=\frac{3}{2}$ then the estimate of the magnetic moment is

$$\mu \;=\; \frac{5.58\left(\frac{3}{2}\times\frac{5}{2}\;+\;\frac{1}{2}\times\frac{3}{2}\;-\;1\times 2\right)+\left(\frac{3}{2}\times\frac{5}{2}\;+\;1\times 2\;-\;\frac{1}{2}\times\frac{3}{2}\right)\frac{3}{2}}{2\times\frac{3}{2}\times\frac{5}{2}}\;=\;3.79\mu_N$$

The measured value is $3.26\mu_N$ so the estimate is not too good. For heavier nuclei the estimate from the shell model gets much worse.

The precise origin of the magnetic dipole moment is not understood, but in general they cannot be predicted from the shell model. For example for the nuclide $^{17}_9$F (fluorine), the measured value of the magnetic moment is $4.72\mu_N$ whereas the value predicted form the above model is $-0.26\mu_N$. !! There are contributions to the magnetic moments from the nuclear potential that is not well-understood.

5.5 Excited States

As in the case of Atomic Physics, nuclei can be in excited states, which decay via the emission of a photon (γ-ray) back to their ground state (either directly ore indirectly).

Some of these excited states are states in which one of the neutrons or protons in the outer shell is promoted to a higher energy level.

However, unlike Atomic Physics, it is also possible that sometimes it is energetically cheaper to promote a nucleon from an inner closed shell, rather than a nucleon form an outer shell into a high energy state. Moreover, excited states in which more than one nucleon is promoted above its ground state is much more common in Nuclear Physics than in Atomic Physics.

Thus the nuclear spectrum of states is very rich indeed, but very complicated and cannot be easily understood in terms of the shell model.

Most of the excited states decay so rapidly that their lifetimes cannot be measured. There are some excited states, however, which are metastable because they cannot decay without violating the selection rules. These excited states are known as “isomers”, and their lifetimes can be measured.

5.6 The Collective Model

The Shell Model has its shortcomings. This is particularly true for heavier nuclei. We have already seen that the Shell Model does not predict magnetic dipole moments or the spectra of excited states very well.

One further failing of the Shell Model are the predictions of electric quadrupole moments, which in the Shell Model are predicted to be very small. However, heavier nuclei with A in the range 150 - 190 and for A $>$ 220, these electric quadrupole moments are found to be rather large.

The failure of the Shell Model to correctly predict electric quadrupole moments arises from the assumption that the nucleons move in a spherically symmetric potential.

The Collective Model generalises the result of the Shell Model by considering the effect of a non-spherically symmetric potential, which leads to substantial deformations for large nuclei and consequently large electric quadrupole moments.

One of the most striking consequences of the Collective Model is the explanation of low-lying excited states of heavy nuclei. These are of two types

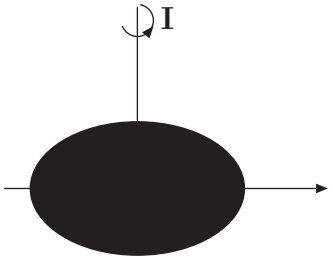
- **Rotational States:** A nucleus whose nucleon density distributions are spherically symmetric (zero quadrupole moment) cannot have rotational excitations (this is analogous to the application of the principle of equipartition of energy to monatomic molecules for which there are no degrees of freedom associated with rotation).

  On the other hand a nucleus with a non-zero quadrupole moment can have excited levels due to rotational perpendicular to the axis of symmetry.

  For an even-even nucleus whose ground state has zero spin, these states have energies

  $$E_{\rm rot} \;=\; \frac{I(I+1)\,\hbar^2}{2\mathcal{I}}, \tag{5.6.2}$$

  where $\mathcal{I}$ is the moment of inertia of the nucleus about an axis through the centre perpendicular to the axis of symmetry.



  It turns out that the rotational energy levels of an even-even nucleus can only take even values of I. For example the nuclide $^{170}_{72}$Hf (hafnium) has a series of rotational states with excitation energies

  $$\text{E (KeV)}: \quad 100, \quad 321, \quad 641$$

These are almost exactly in the ratio $2 \times 3 : \ 4 \times 5 : \ 6 \times 7$, meaning that these are states with rotational spin equal to 2, 4, 6 respectively. The relation is not exact because the moment of inertia changes as the spin increases.

We can extract the moment of inertia for each of these rotational states from eq.(5.6.2). We could express this in SI units, but more conveniently nuclear moments of inertia are quoted in $\mathrm{MeV/c^2\,fm^2}$, with the help of the relation

$$\hbar\, c \ = \ 197.3\,\mathrm{MeV\,fm}.$$

Therefore the moment inertia of the $I = 2$ state, whose excitation energy is 0.1 MeV, is given (inseting $I = 2$ into eq.(5.6.2) by

$$\mathcal{I} \ = 2 \times 3 \times \frac{\hbar^2 c^2}{2c^2 E_{\mathrm{rot}}} \ = \ \frac{6}{2}\frac{197.3^2}{0.1} \ = \ 1.17 \times 10^6\,\mathrm{MeV/c^2\,fm^2}$$

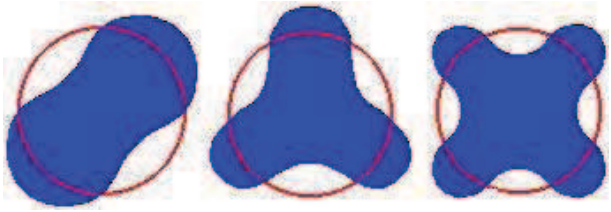
For odd-A nuclides for which the spin of the ground state I_0 is non-zero, the rotational levels have excitation levels of

$$E_{\mathrm{rot}} \ = \ \frac{1}{2\mathcal{I}}\left(I(I+1) \ - \ I_0(I_0+1)\right)\hbar^2,$$

where I can take the values $I_0 + 1, \ I_0 + 2$ etc. For example the first two rotational excitation energies of ${}^{143}_{60}\mathrm{Nd}$ (neodynium), whose ground state has spin $\frac{7}{2}$, have energies 128 KeV and 290 KeV. They correspond to rotational levels with nuclear spin $\frac{9}{2}$ and $\frac{11}{2}$ respectively. The ratio of these two excitation energies (2.27) is almost exactly equal to

$$\frac{\frac{11}{2} \times \frac{13}{2} - \frac{7}{2} \times \frac{9}{2}}{\frac{9}{2} \times \frac{11}{2} - \frac{7}{2} \times \frac{9}{2}} \ = \ 2.22$$

- **Shape oscillations:** These are modes of vibration in which the deformation of the nucleus oscillates - the electric quadrupole moment oscillates about its mean value. It could be that this mean value is very small, in which case the nucleus is oscillating between an oblate and a prolate spheroidal shape. It is also possible to have shape oscillations with different shapes



The small oscillations about the equilibrium shape perform simple harmonic motion. The energy levels of such modes are equally spaced. Thus an observed sequence of equally spaced energy levels within the spectrum of a nuclide is interpreted as a manifestation of such shape oscillations.