Chapter 4. Electrostatic Fields in Matter

4.1. Polarization

A neutral atom, placed in an external electric field, will experience no net force. However, even though the atom as a whole is neutral, the positive charge is concentrated in the nucleus (radius = 10^{-14} m) while the negative charge forms an electron cloud (radius = 10^{-10} m) surrounding the nucleus (see Figure 4.1). The nucleus of the atom will experience a force pointing in the same direction as the external electric field (to the right in Figure 4.1) and of magnitude qE_{ext}. The negatively charged electron cloud will experience a force of the same magnitude, but pointed in a direction opposite to the direction of the electric field. As a result of the external force, the nucleus will move in the direction of the electric field until the external force on it is canceled by the force exerted on the nucleus by the electron cloud.

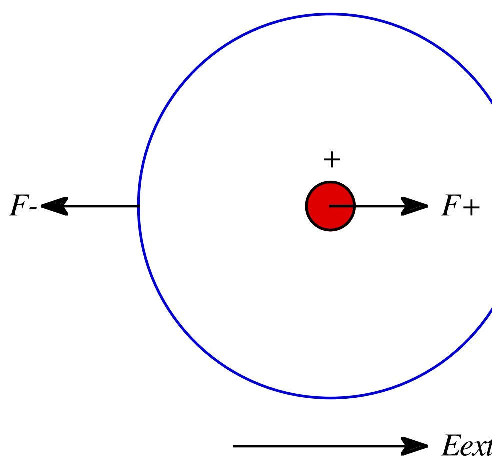

Figure 4.1. Atom in external electric field.

Consider an electron cloud with a constant volume charge density ρ and a radius a. If the total charge of the electron cloud is $-q$ then the corresponding charge density ρ is equal to

$$\rho = \frac{-q}{\frac{4}{3}\pi a^3} = -\frac{3q}{4\pi a^3}$$

The electric field inside the uniformly charged cloud is equal to

$$E(r) = -\frac{1}{4\pi\varepsilon_0}\frac{qr}{a^3}$$

where r is the distance from the center of the cloud. Suppose that as a result of the external electric field the nucleus moves by a distance d with respect to the center of the electron cloud. The electric force exerted on the nucleus by the electron cloud is equal to

$$F_{cloud} = qE(d) = -\frac{1}{4\pi\varepsilon_0}\frac{q^2 d}{a^3}$$

The equilibrium position of the nucleus is that position where the external force is canceled by the force exerted on it by the electron cloud:

$$\overline{F}_{cloud} + \overline{F}_{ext} = 0$$

This expression can be rewritten as

$$qE_{ext} - \frac{1}{4\pi\varepsilon_0}\frac{q^2 d}{a^3} = 0$$

The equilibrium distance d is thus equal to

$$d = 4\pi\varepsilon_0 a^3 \frac{E_{ext}}{q}$$

The induced dipole moment $\overline{p}$ of the atom is defined as

$$\overline{p} = q\overline{d} = 4\pi\varepsilon_0 a^3 \overline{E}_{ext}$$

Therefore, the magnitude of the induced dipole moment is proportional to the magnitude of the external electric field, and its direction is equal to the direction of the external electric field. The constant of proportionality is called the **atomic polarizability** α and is defined as

$$\alpha = \frac{|\overline{p}|}{|\overline{E}_{ext}|} = 4\pi\varepsilon_0 a^3$$

Although this model of the atom is extremely crude, it produces results that are in reasonable agreement with direct measurements of the atomic polarizability.

Example: Problem 4.2

According to quantum mechanics, the electron cloud for a hydrogen atom in its ground state has a charge density equal to

$$\rho(r) = \frac{q}{\pi a^3} e^{-2r/a}$$

where q is the charge of the electron and a is the Bohr radius. Find the atomic polarizability of such an atom.

As a result of an external electric field the nucleus of the atom will be displaced by a distance d with respect to the center of the electron cloud. The force exerted on the nucleus by the electron cloud is equal to

$$\overline{F}_{cloud}(d) = q\overline{E}_{cloud}(d)$$

where $\overline{E}_{cloud}$ is the electric field generated by the electron cloud. The electric field generated by the electron cloud can be calculated using Gauss's law:

$$\overline{E}_{cloud}(d) = \frac{1}{4\pi d^2}\frac{Q_{encl}}{\varepsilon_0} = \frac{1}{\varepsilon_0 d^2}\int_0^d \frac{q}{\pi a^3} e^{-2r/a} r^2 dr = \frac{1}{4\pi\varepsilon_0}\frac{q}{d^2}\left[1 - e^{-2d/a}\left(1 + 2\frac{d}{a} + 2\frac{d^2}{a^2}\right)\right]$$

The displacement of the nucleus will be very small compared to the size of the electron cloud (d«a). Therefore, we can expand exp(-2d/a) in terms of d/a:

$$\overline{E}_{cloud}(d) = \frac{1}{4\pi\varepsilon_0}\frac{q}{d^2}\left[1 - \left(1 - 2\frac{d}{a} + 2\left(\frac{d}{a}\right)^2 - \frac{4}{3}\left(\frac{d}{a}\right)^3 \ldots\right)\left(1 + 2\frac{d}{a} + 2\frac{d^2}{a^2}\right)\right] \cong$$

$$\cong \frac{1}{4\pi\varepsilon_0}\frac{q}{d^2}\frac{4}{3}\left(\frac{d}{a}\right)^3 = \frac{1}{3\pi\varepsilon_0}\frac{qd}{a^3} = \frac{1}{3\pi\varepsilon_0}\frac{p}{a^3}$$

The nucleus will be in an equilibrium position when the electric force exerted on it by the external field is equal to the electric force exerted on it by the electron cloud. This occurs when the electric field at the position of the nucleus, generated by he electron cloud, is equal in magnitude to the externally applied electric field, but pointing in the opposite direction. The dipole moment of the dipole can therefore be expressed in terms of the external field:

$$\overline{p} = 3\pi\varepsilon_0 a^3 \overline{E}_{ext}$$

The electric polarizability of the material is therefore equal to

$$\alpha = \frac{|\bar{p}|}{|\bar{E}_{ext}|} = 3\pi\varepsilon_0 a^3$$

which is close to the result obtained using the classical model of the atom.

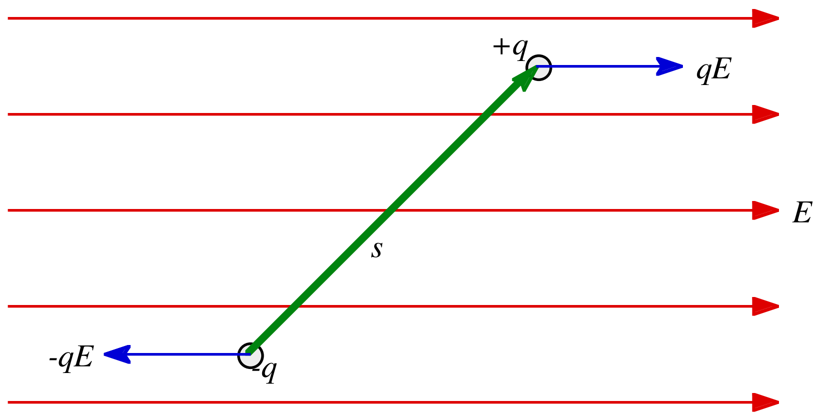

Figure 4.2. Torque on dipole in an electric field.

Besides polarizing the atoms of a material, the external electric field can align its molecules. Some molecules, like water, have a permanent dipole moment. Normally, the dipole moments of the water molecules will be directed randomly, and the average dipole moment is zero. When the water is exposed to an external electric field, a torque is exerted on the water molecule, and it will try to align its dipole moment with the external electric field. This is schematically illustrated in Figure 4.2. Figure 4.2 shows a dipole $\bar{p} = q\bar{s}$ placed in an electric field, directed along the x axis. The net force on the dipole is zero since the net charge is equal to zero. The torque on the dipole with respect to its center is equal to

$$\bar{N} = \left(\bar{r}_+ \times \bar{F}_+\right) + \left(\bar{r}_- \times \bar{F}_-\right) = \left(\left(\frac{1}{2}\bar{s}\right) \times \left(q\bar{E}\right)\right) + \left(\left(-\frac{1}{2}\bar{s}\right) \times \left(-q\bar{E}\right)\right) = q\bar{s} \times \bar{E} = \bar{p} \times \bar{E}$$

As a result of this torque, the dipole will try to align itself with the electric field. When the dipole moment is pointing in the same direction as the electric field the torque on the dipole will be equal to zero.

Example: Problem 4.6

A dipole with dipole moment $\bar{p}$ is situated a distance d above an infinite grounded conducting plane (see Figure 4.3). The dipole makes and angle θ with the perpendicular to the plane. Find the torque on $\bar{p}$. If the dipole is free to rotate, in what direction will it come to rest?

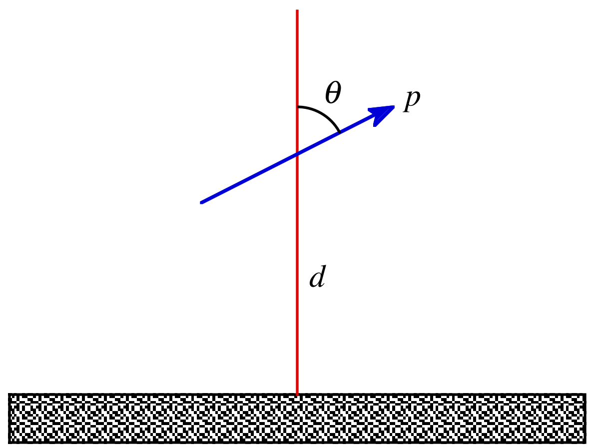

Figure 4.3. Problem 4.6

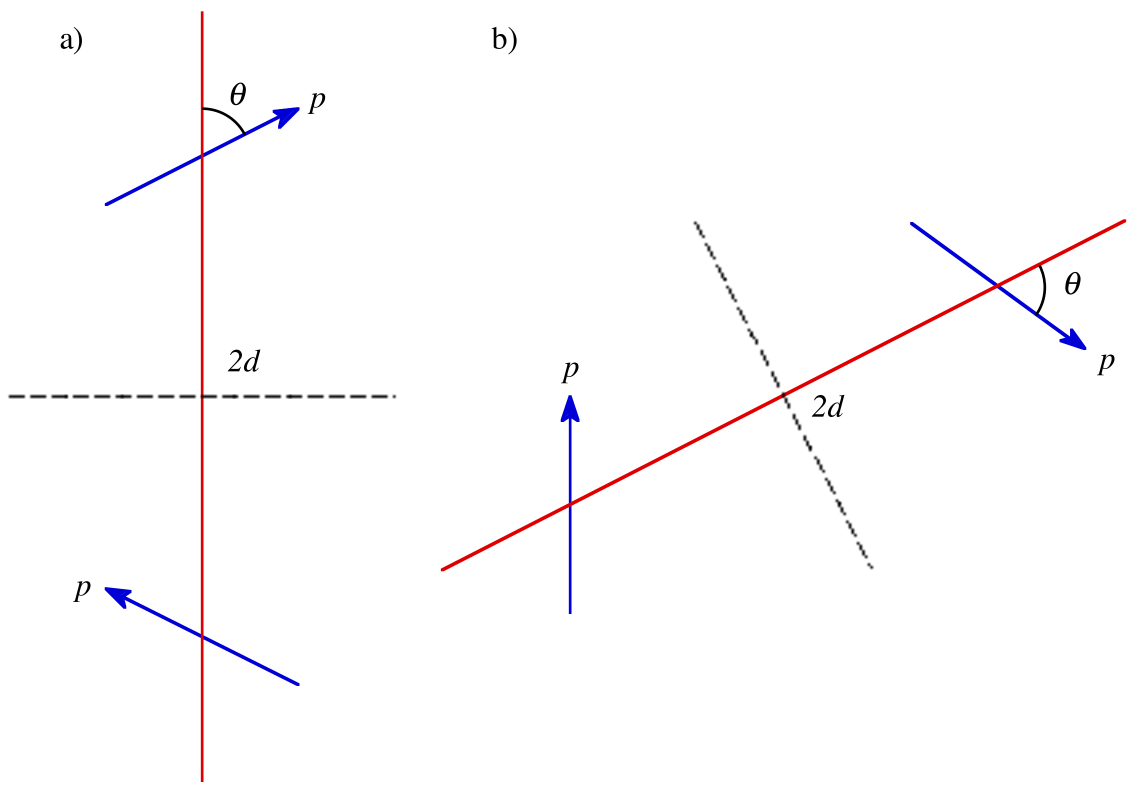

Figure 4.4. Method of images (Problem 4.6).

This problem can be solved using the method of images (see Figure 4.4a). Note that the method of images, when applied to a dipole, does not produce an exact mirror image of the dipole. After defining the image dipole, we chose a new coordinate system such that the image dipole is located at the origin, and pointing upwards (along the positive z axis, see Figure 4.4b). The electric field at the position of the real dipole due to the image dipole is equal to

$$\overline{E}_{image} = \frac{1}{4\pi\varepsilon_0}\frac{1}{(2d)^3}\left\{3\left(\overline{p}_{image} \bullet \hat{r}\right)\hat{r} - \overline{p}_{image}\right\} = \frac{1}{32\pi\varepsilon_0}\frac{1}{d^3}\left\{3p\cos\theta\,\hat{r} - p\,\hat{k}\right\}$$

The torque on the real dipole is equal to

$$\overline{N} = \overline{p}\times\overline{E}_{image} = \frac{1}{32\pi\varepsilon_0}\frac{1}{d^3}\left\{3p\cos\theta\left(\overline{p}\times\hat{r}\right) - p\left(\overline{p}\times\hat{k}\right)\right\} =$$

$$= \frac{1}{32\pi\varepsilon_0}\frac{1}{d^3}\left\{3p^2\cos\theta\sin\theta\,\hat{j} - p^2\sin 2\theta\,\hat{j}\right\} = \frac{1}{64\pi\varepsilon_0}\frac{1}{d^3}p^2\sin 2\theta\,\hat{j}$$

The torque on the dipole is positive when $0 < \theta < \pi/2$ and as a consequence the dipole will rotate counter clockwise towards the stable orientation of $\theta = 0$. The torque on the dipole is negative when $\pi/2 < \theta < \pi$ and as a consequence the dipole will rotate clockwise towards the stable orientation of $\theta = \pi$.

Example: Problem 4.7

Show that the energy of a dipole in an electric field is given by

$$U = -\overline{p} \bullet \overline{E}$$

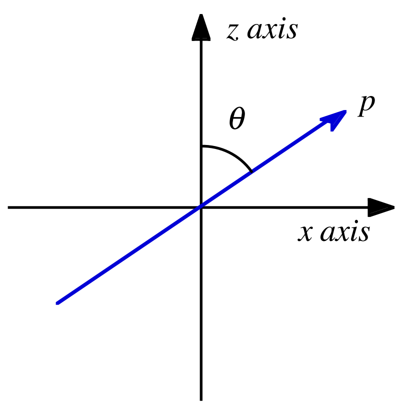

Figure 4.5. Problem 4.7.

Consider the dipole located at the origin of a coordinate system. The z axis of the coordinate system coincides with the direction of the electric field and the angle between the dipole and the z axis is equal to θ (see Figure 4.5). The energy of the system can be determined by calculating the work to be done to move the dipole from infinity to its present location. Assume the dipole is initially oriented parallel to the x axis and is first moved from infinity along the x axis to $r = 0$. The force exerted on the dipole by the electric field is directed perpendicular to the displacement and therefore the work done by this force is equal to zero. The dipole is then rotated to its final position (from $\pi/2$ to θ). The torque exerted by the electric field on the dipole is equal to

$$\overline{N} = \overline{p} \times \overline{E} = pE \sin\theta \, \hat{k}$$

In order to rotate the dipole I must supply a torque opposite to $\overline{N}$:

$$\overline{N}_{\text{supplied}} = -pE \sin\theta \, \hat{k}$$

Therefore, the work done by me is equal to

$$W = -\int_{\pi/2}^{\theta} N_{\text{supplied}} \, d\theta = \int_{\pi/2}^{\theta} pE \sin\theta d\theta = -pE \cos\theta = -\overline{p} \bullet \overline{E}$$

The potential energy of the dipole is therefore equal to

$$U = -\overline{p} \bullet \overline{E}$$

and reaches a minimum when $\overline{p}$ is parallel to $\overline{E}$ (the dipole is aligned with the electric field).

4.2. The Field of a Polarized Object

Consider a piece of polarized material with a dipole moment per unit volume equal to $\overline{P}$. The electrostatic potential generated by this material is equal to

$$V(\overline{r}) = \frac{1}{4\pi\varepsilon_0} \int_{Volume} \frac{\Delta\hat{r} \bullet \overline{P}}{(\Delta r)^2} d\tau = \frac{1}{4\pi\varepsilon_0} \int_{Volume} \overline{P} \bullet \overline{\nabla}\left(\frac{1}{\Delta r}\right) d\tau$$

where $\Delta\overline{r} = \overline{r} - \overline{r}\,'$. Using the following relation (one of the product rules of the vector operator)

$$\overline{\nabla} \bullet \left(\frac{1}{\Delta r}\overline{P}\right) = \frac{1}{\Delta r}\left(\overline{\nabla} \bullet \overline{P}\right) + \overline{P} \bullet \overline{\nabla}\left(\frac{1}{\Delta r}\right)$$

we can rewrite the expression for the electric potential as

$$V(\bar{r}) = \frac{1}{4\pi\varepsilon_0}\int_{Volume} \bar{\nabla}\bullet\left(\frac{1}{\Delta r}\bar{P}\right)d\tau - \frac{1}{4\pi\varepsilon_0}\int_{Volume}\frac{1}{\Delta r}\left(\bar{\nabla}\bullet\bar{P}\right)d\tau =$$

$$= \frac{1}{4\pi\varepsilon_0}\int_{Surface}\frac{1}{\Delta r}\bar{P}\bullet d\bar{a} - \frac{1}{4\pi\varepsilon_0}\int_{Volume}\frac{1}{\Delta r}\left(\bar{\nabla}\bullet\bar{P}\right)d\tau =$$

$$= \frac{1}{4\pi\varepsilon_0}\int_{Surface}\frac{1}{\Delta r}\sigma_b da + \frac{1}{4\pi\varepsilon_0}\int_{Volume}\frac{1}{\Delta r}\rho_b d\tau$$

where

$$\sigma_b = \bar{P}\bullet d\hat{n} \text{ (bound surface charge)}$$

and

$$\rho_b = -\left(\bar{\nabla}\bullet\bar{P}\right) \text{ (bound volume charge)}$$

Here the unit vector $\hat{n}$ is perpendicular to the integration surface (and pointing outwards). The equation for the electrostatic potential shows that the potential (and therefore also the electric field) generated by a polarized object is equal to the potential generated by an object with surface charge density σ_b and volume charge density ρ_b.

Example: Problem 4.10

A sphere of radius *R* carries a polarization

$$\bar{P}(\bar{r}) = k\bar{r}$$

where *k* is a constant and $\bar{r}$ is the vector from the center.

a) Calculate the bound charges σ_b and ρ_b.

b) Find the field inside and outside the sphere.

a) The unit vector $\hat{n}$ on the surface of the sphere is equal to the radial unit vector. The bound surface charge is equal to

$$\sigma_b = \bar{P}\bullet\hat{n}\Big|_{r=R} = k\bar{r}\bullet\hat{r}\Big|_{r=R} = kR$$

The bound volume charge is equal to

$$\rho_b = -\left(\bar{\nabla}\bullet\bar{P}\right) = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 kr\right) = -3k$$

b) First consider the region outside the sphere. The electric field in this region due to the surface charge is equal to

$$\bar{E}_{surface}(r) = \frac{1}{4\pi\varepsilon_0}\frac{4\pi R^2\sigma_b}{r^2}\hat{r} = \frac{kR^3}{\varepsilon_0 r^2}\hat{r}$$

The electric field in this region due to the volume charge is equal to

$$\bar{E}_{volume}(r) = \frac{1}{4\pi\varepsilon_0}\frac{\frac{4}{3}\pi R^3\rho_b}{r^2}\hat{r} = -\frac{kR^3}{\varepsilon_0 r^2}\hat{r}$$

Therefore, the total electric field outside the sphere is equal to zero.

Now consider the region inside the sphere. The electric field in this region due to the surface charge is equal to zero. The electric field due to the volume charge is equal to

$$\bar{E}_{volume}(r) = \frac{1}{4\pi\varepsilon_0}\frac{\frac{4}{3}\pi r^3\rho_b}{r^2}\hat{r} = -\frac{kr}{\varepsilon_0}\hat{r}$$

The bound charges introduced in this Section are not just mathematical artifacts, but are real charges, bound to the individual dipoles of the material. Consider for example the three dipoles shown in Figure 4.6a. When they are aligned (lengthwise) the center charges cancel, and the system looks like a single dipole with dipole moment $3dq$ (see Figure 4.6b).

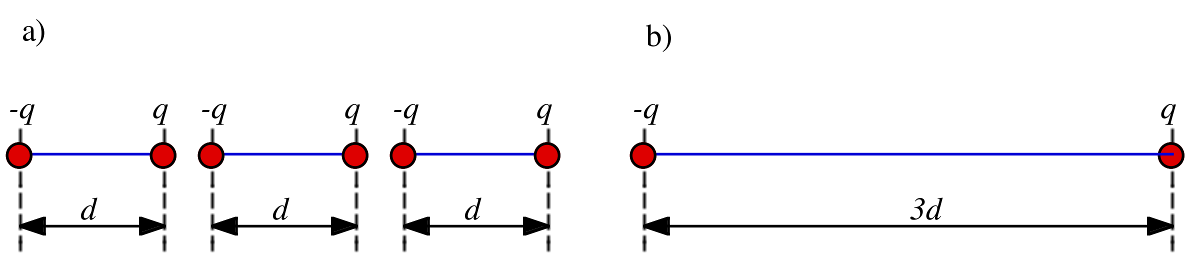

Figure 4.6. Aligned dipoles.

In a uniformly polarized material of thickness s and polarization $\bar{P}$ all dipoles are perfectly aligned (see Figure 4.7). The net result of the alignment of the individual dipoles is a positive surface charge on one side of the material and negative surface charge on the opposite side. Consider a cylinder with surface area A whose axis is aligned with the direction of polarization of the polarized material. The total dipole moment of this cylinder is equal to

$$\bar{p}_{cylinder} = As\bar{P}$$

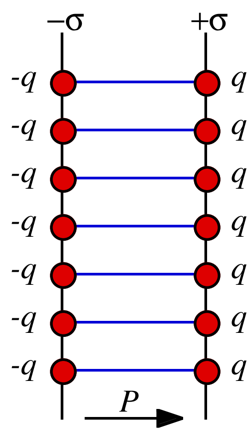

Figure. 4.7. Uniform polarization.

Since the only charge of the system resides on the end caps of the cylinder (volume charges cancel in a uniformly polarized material: see Figure 4.6), the net charge there must be equal to

$$q_{end} = \frac{p_{cylinder}}{s} = AP$$

The charge density on the surface is therefore equal to

$$\sigma = \frac{q_{end}}{A} = P$$

If the surface of the material is not perpendicular to the direction of polarization then surface charge density will be less than P (surface charge distributed over a larger area) and equal to

$$\sigma = \bar{P} \bullet \hat{n}$$

where $\hat{n}$ is the unit vector perpendicular to the surface of the material, pointing outwards. For the material shown in Figure 4.7 this equation immediately shows that a positive surface charge resides on the right surface ($\bar{P}$ parallel to $\hat{n}$) and a negative surface charge resides on the left surface ($\bar{P}$ anti parallel to $\hat{n}$). Since these charges reside on the surface and are bound to the dipoles they are called the **bound surface charge** or σ_b.

If the material is uniformly polarized then the volume charge density is equal to zero (see Figure 4.6). However, if the polarization is not uniform then there will be a net volume charge inside the material. Consider a system of three aligned dipoles (see Figure 4.8). If the polarization is not uniform then the strength of the individual dipoles will vary. Assuming that the physical size (length) of the dipoles shown in Figure 4.8 is the same, then the varying dipole strength is a result of variations in the charge on the ends of the dipoles. Since the net charge on the polarized material must be equal to zero, the sum of the volume charges and surface charges must be equal to zero. Thus

$$\int_{Surface} \sigma_b \, da + \int_{Volume} \rho_b \, d\tau = 0$$

This equation can be rewritten by substituting the expression for the surface charge density and applying the fundamental theorem of divergences:

$$\int_{Volume} \rho_b \, d\tau = -\int_{Surface} \sigma_b \, da = -\int_{Surface} \overline{P} \bullet d\overline{a} = -\int_{Volume} \left(\overline{\nabla} \bullet \overline{P}\right) d\tau$$

Since this relation holds for any volume we can conclude that

$$\rho_b = -\left(\overline{\nabla} \bullet \overline{P}\right)$$

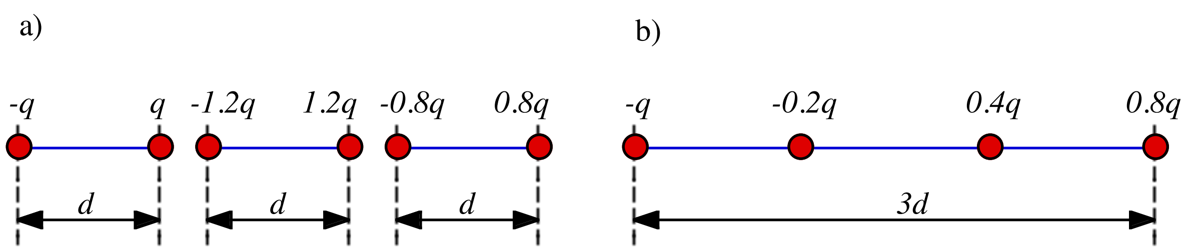

Figure 4.8. Non-uniform polarization.

Example: Problem 4.31

A dielectric cube of side s, centered at the origin, carries a "frozen-in" polarization $\overline{P} = k\overline{r}$, where k is a constant. Find all the bound charges, and check that they add up to zero.

The bound volume charge density ρ_b is equal to

$$\rho_b = -\left[\overline{\nabla} \bullet \overline{P}\right] = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 kr\right) = -3k$$

Since the bound volume charge density is constant, the total bound volume charge in the cube is equal to product of the charge density and the volume:

$$q_{volume} = -3ks^3$$

The surface charge density σ_b is equal to

$$\sigma_b = \overline{P} \bullet \hat{n} = k\overline{r} \bullet \hat{n}$$

The scalar product between $\overline{r}$ and $\hat{n}$ can be evaluate easily (see Figure 4.9) and is equal to

$$\overline{r} \bullet \hat{n} = r\cos\theta = \frac{1}{2}s$$

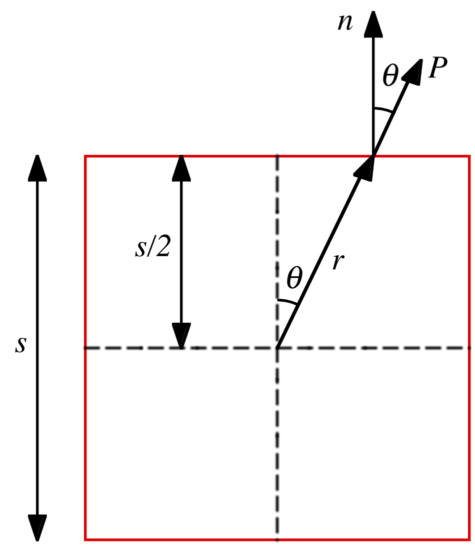

Figure 4.9. Problem 4.31.

Therefore the surface charge density is equal to

$$\sigma_b = k\overline{r} \bullet \hat{n} = \frac{1}{2}ks$$

The surface charge density is constant across the surface of the cube and consequently the total surface charge on the cube is equal to the product of the surface charge density and the total surface area of the cube:

$$q_{surface} = \left(\frac{1}{2}ks\right)\left(6s^2\right) = 3ks^3$$

The total bound charge on the cube is equal to

$$q_{total} = q_{volume} + q_{surface} = -3ks^3 + 3ks^3 = 0$$

4.3. The Electric Displacement

The electric field generated by a polarized material is equal to the electric field produced by its bound charges. If free charges are also present then the total electric field produced by this system is equal to the vector sum of the electric fields produced by the bound charges and by the free charges. Gauss's law can also be used for this type of systems to calculate the electric field as long as we include both free and bound charges:

$$\overline{\nabla} \bullet \overline{E} = \frac{\rho}{\varepsilon_0} = \frac{\rho_{bound} + \rho_{free}}{\varepsilon_0} = \frac{1}{\varepsilon_0}\left(-\overline{\nabla} \bullet \overline{P} + \rho_{free}\right)$$

where $\overline{P}$ is the polarization of the material. This expression can be rewritten as

$$\overline{\nabla} \bullet \left(\varepsilon_0 \overline{E} + \overline{P}\right) = \rho_{free}$$

The expression in parenthesis is called the **electric displacement** $\overline{D}$ which is defined as

$$\overline{D} = \varepsilon_0 \overline{E} + \overline{P}$$

In terms of $\overline{D}$, Gauss's law can be rewritten as

$$\overline{\nabla} \bullet \overline{D} = \rho_{free} \qquad \text{(Gauss's law in differential form)}$$

and

$$\oint_{Surface} \overline{D} \bullet d\overline{a} = Q_{free} \qquad \text{(Gauss's law in integral form)}$$

These two versions of Gauss's law are particularly useful since they make reference only to free charges, which are the charges we can control.

Although it seems that the displacement $\overline{D}$ has properties similar to the electric field $\overline{E}$ there are some very significant differences. For example, the curl of $\overline{D}$ is equal to

$$\overline{\nabla} \times \overline{D} = \varepsilon_0 \overline{\nabla} \times \overline{E} + \overline{\nabla} \times \overline{P} = \overline{\nabla} \times \overline{P}$$

and is in general not equal to zero. Since the curl of $\overline{D}$ is not necessarily equal to zero, there is in general no potential that generates $\overline{D}$.

The Helmholtz theorem tell us that if we know the curl and the divergence of a vector function $\bar{v}$ then this is sufficient information to uniquely define the vector function $\bar{v}$. Therefore, the electric field $\bar{E}$ is uniquely defined by Gauss's law since we know that he curl of $\bar{E}$ is zero, everywhere. The displacement current $\bar{D}$ on the other hand is not uniquely determined by the free charge distribution, but requires additional information (like for example $\bar{P}$).

Example: Problem 4.16

Suppose the field inside a large piece of dielectric is $\bar{E}_0$, so that the electric displacement is equal to $\bar{D}_0 = \varepsilon_0 \bar{E}_0 + \bar{P}$.

a) Now, a small spherical cavity is hollowed out of the material. Find the field at the center of the cavity in terms of $\bar{E}_0$ and $\bar{P}$. Also find the displacement at the center of the cavity in terms of $\bar{D}_0$ and $\bar{P}$.

b) Do the same for a long needle-shaped cavity running parallel to $\bar{P}$.

c) Do the same for a thin wafer-shaped cavity perpendicular to $\bar{P}$.

a) Consider a large piece of dielectric material with polarization $\bar{P}$ and a small sphere with polarization $-\bar{P}$ superimposed on it. The field generated by this system is equal to the field generated by the dielectric material with a small spherical cavity hollowed out (principle of superposition). The electric field inside a sphere with polarization $-\bar{P}$ is uniform and equal to

$$\bar{E}_{sphere} = -\frac{1}{3\varepsilon_0}\left(-\bar{P}\right) = \frac{1}{3\varepsilon_0}\bar{P}$$

(see Example 2 of Griffiths). The field at the center of the cavity is therefore equal to

$$\bar{E}_{center} = \bar{E}_0 + \bar{E}_{sphere} = \bar{E}_0 + \frac{1}{3\varepsilon_0}\bar{P}$$

The corresponding electric displacement at the center of the cavity is equal to

$$\bar{D}_{center} = \varepsilon_0 \bar{E}_{center} = \varepsilon_0 \bar{E}_0 + \frac{1}{3}\bar{P} = \bar{D}_0 - \frac{2}{3}\bar{P}$$

b) Consider a large piece of dielectric material with polarization $\bar{P}$ and a small long needle-shaped piece with polarization $-\bar{P}$ superimposed on it. The field generated by this system is equal to the field generated by the dielectric material with a small long needle-shaped cavity hollowed out (principle of superposition). The electric field of a polarized needle of length s is equal to that of two point charges (+q and -q) located a distance s apart. The charge on top of the needle will be negative, while the charge on the bottom of the needle will be positive. The charge density on the end caps of the needle is equal to P. Therefore,

$$q = \sigma_b A = PA$$

where A is the surface area of the end caps of the needle. The electric field generated by the needle at its center is equal to

$$\overline{E}_{needle} = \frac{1}{4\pi\varepsilon_0}\frac{(+PA)}{\frac{1}{4}s^2}\hat{k} - \frac{1}{4\pi\varepsilon_0}\frac{(-PA)}{\frac{1}{4}s^2}\hat{k} = \frac{2}{\pi\varepsilon_0}\frac{PA}{s^2}\hat{k}$$

In the needle limit $A \to 0$ and therefore $\overline{E}_{needle} \to 0$. Thus at the center of the needle cavity

$$\overline{E}_{center} = \overline{E}_0$$

The electric displacement at this point is equal to

$$\overline{D}_{center} = \varepsilon_0 \overline{E}_{center} = \varepsilon_0 \overline{E}_0 = \overline{D}_0 - \overline{P}$$

c) Consider a large piece of dielectric material with polarization $\overline{P}$ and a thin wafer-shaped piece of dielectric material with polarization $-\overline{P}$ superimposed on it. The field generated by this system is equal to the field generated by the dielectric material with a thin wafer-shaped cavity hollowed out (principle of superposition). The electric field inside the wafer will be that of two parallel plates with charge densities equal to -σ on the top and +σ on the bottom. For a thin wafer-shaped cavity the electric field between the plates will be equal to the field of a parallel-plate capacitor with infinitely large plates. Thus

$$\overline{E}_{wafer} = \frac{\sigma}{\varepsilon_0}\hat{k} = \frac{1}{\varepsilon_0}\overline{P}$$

The net electric field in the center of the cavity is therefore equal to

$$\overline{E}_{center} = \overline{E}_0 + \overline{E}_{wafer} = \overline{E}_0 + \frac{1}{\varepsilon_0}\overline{P}$$

The electric displacement at the center of the cavity is equal to

$$\overline{D}_{center} = \varepsilon_0 \overline{E}_{center} = \varepsilon_0 \overline{E}_0 + \overline{P} = \overline{D}$$

4.4. Linear Dielectrics

Most dielectric materials become polarized when they are placed in an external electric field. In many materials the polarization is proportional to the electric field:

$$\overline{P} = \varepsilon_0 \chi_e \overline{E}$$

where $\overline{E}$ is the **total** electric field (external + internal). The constant of proportionality, χ_e, is called the **electric susceptibility**. Materials in which the induced polarization is proportional to the electric field are called **linear dielectrics**.

The electric displacement in a linear dielectric is also proportional to the total electric field:

$$\overline{D} = \varepsilon_0 \overline{E} + \overline{P} = \varepsilon_0 \left(1 + \chi_e\right) \overline{E} = \varepsilon \overline{E}$$

where ε is called the **permittivity** of the material which is equal to

$$\varepsilon = \varepsilon_0 \left(1 + \chi_e\right)$$

The constant $1 + \chi_e$ is called the **dielectric constant** *K* of the material.

Consider a volume *V* entirely filled with linear dielectric material with dielectric constant *K*. The polarization $\overline{P}$ of this material is equal to

$$\overline{P} = \varepsilon_0 \chi_e \overline{E}$$

and is therefore proportional to $\overline{E}$ everywhere. Therefore

$$\overline{\nabla} \times \overline{P} = \varepsilon_0 \chi_e \left(\overline{\nabla} \times \overline{E}\right) = 0$$

and consequently

$$\overline{\nabla} \times \overline{D} = \varepsilon_0 \left(\overline{\nabla} \times \overline{E}\right) + \overline{\nabla} \times \overline{P} = 0$$

The electric displacement $\overline{D}$ therefore satisfies the following two conditions:

$$\overline{\nabla} \times \overline{D} = 0$$

and

$$\overline{\nabla} \bullet \overline{D} = \rho_{free}$$

The electric field generated by the free charges when the dielectric is not present satisfies the following two equations:

$$\overline{\nabla} \times \overline{E}_{free} = 0$$

and

$$\overline{\nabla} \bullet \overline{E}_{free} = \frac{\rho_{free}}{\varepsilon_0}$$

Comparing the two sets of differential equations for $\overline{D}$ and $\overline{E}_{free}$ we conclude that

$$\overline{D} = \varepsilon_0 \overline{E}_{free}$$

The displacement $\overline{D}$ can also be expressed in terms of the total field inside the dielectric:

$$\overline{D} = \varepsilon_0 (1 + \chi_e) \overline{E} = \varepsilon \overline{E}$$

These two equations show that

$$\overline{E} = \frac{\varepsilon_0}{\varepsilon} \overline{E}_{free} = \frac{1}{K} \overline{E}_{free}$$

The presence of the dielectric material therefore reduces the electric field by a factor K.

Example: Problem 4.18

The space between the plates of a parallel-plate capacitor (see Figure 4.10) is filled with two slabs of linear dielectric material. Each slab has thickness s, so that the total distance between the plates is $2s$. Slab 1 has a dielectric constant of 2, and slab 2 has a dielectric constant of 1.5. The free charge density on the top plate is σ and on the bottom plate is $-\sigma$.

a) Find the electric displacement $\overline{D}$ in each slab.
b) Find the electric field $\overline{E}$ in each slab.
c) Find the polarization $\overline{P}$ in each slab.
d) Find the potential difference between the plates.
e) Find the location and amount of all bound charge.
f) Now that you know all charges (free and bound), recalculate the field in each slab, and compare with your answers to b).

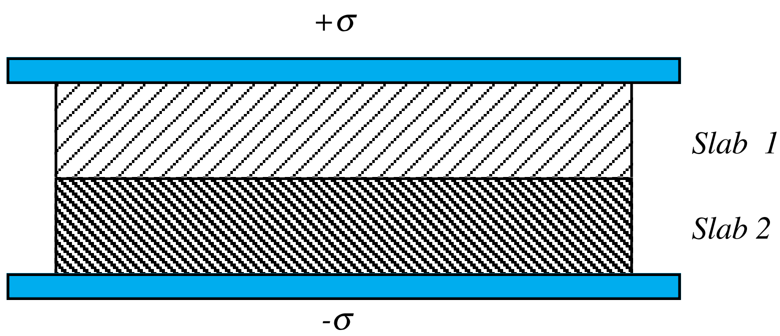

Figure 4.10. Problem 4.18.

a) The electric displacement $\overline{D}_1$ in slab 1 can be calculated using "Gauss's law". Consider a cylinder with cross sectional area A and axis parallel to the z axis, being used as a Gaussian surface. The top of he cylinder is located inside the top metal plate (where the electric displacement is zero) and the bottom of the cylinder is located inside the dielectric of slab 1. The electric displacement is directed parallel to the z axis and pointed downwards. Therefore, the displacement flux through this surface is equal to

$$\Phi_D = D_1 A$$

The free charge enclosed by this surface is equal to

$$Q_{free,encl} = \sigma A$$

Combining these two equations we obtain

$$D_1 = \frac{\Phi_D}{A} = \frac{Q_{free,encl}}{A} = \sigma$$

In vector notation

$$\overline{D}_1 = -\sigma \hat{k}$$

In the same way we obtained for slab 2

$$\overline{D}_2 = -\sigma \hat{k}$$

b) The electric field $\overline{E}_1$ in slab 1 is equal to

$$\overline{E}_1 = \frac{1}{K_1 \varepsilon_0} \overline{D}_1 = -\frac{\sigma}{K_1 \varepsilon_0} \hat{k} = -\frac{\sigma}{2\varepsilon_0} \hat{k}$$

The electric field $\overline{E}_2$ in slab 2 is equal to

$$\overline{E}_2 = \frac{1}{K_2\varepsilon_0}\overline{D}_2 = -\frac{\sigma}{K_2\varepsilon_0}\hat{k} = -\frac{2\sigma}{3\varepsilon_0}\hat{k}$$

c) Once $\overline{D}$ and $\overline{E}$ are known, the polarization $\overline{P}$ can be calculated:

$$\overline{P} = \overline{D} - \varepsilon_0\overline{E}$$

Therefore, the polarization of slab 1 is equal to

$$\overline{P}_1 = \overline{D}_1 - \varepsilon_0\overline{E}_1 = -\sigma\hat{k} + \frac{\sigma}{2}\hat{k} = -\frac{\sigma}{2}\hat{k}$$

The polarization of slab 2 is equal to

$$\overline{P}_2 = \overline{D}_2 - \varepsilon_0\overline{E}_2 = -\sigma\hat{k} + \frac{2\sigma}{3}\hat{k} = -\frac{\sigma}{3}\hat{k}$$

d) The potential difference between the top plate and the bottom plate is equal to

$$\Delta V = V_{top} - V_{bottom} = -\int_{bottom}^{top} \overline{E} \bullet d\overline{l} = E_1 s + E_2 s = \left(\frac{\sigma}{2\varepsilon_0} + \frac{2\sigma}{3\varepsilon_0}\right)s = \frac{7\sigma}{6\varepsilon_0}$$

e) There are no bound volume charges (constant polarization). The bound surface charge density on the surface of a dielectric with polarization $\overline{P}$ is equal to $\overline{P} \bullet \hat{n}$. For slab 1 the polarization is equal to

$$\overline{P}_1 = -\frac{\sigma}{2}\hat{k}$$

The surface charge density on the top of slab 1 is equal to

$$\sigma_{top,1} = \overline{P}_1 \bullet \hat{n} = \left(-\frac{\sigma}{2}\hat{k}\right) \bullet \hat{k} = -\frac{\sigma}{2}$$

The surface charge density on the bottom of slab 1 is equal to

$$\sigma_{bottom,1} = \overline{P}_1 \bullet \hat{n} = \left(-\frac{\sigma}{2}\hat{k}\right) \bullet \left(-\hat{k}\right) = \frac{\sigma}{2}$$

For slab 2 the polarization is equal to

$$\overline{P}_2 = -\frac{\sigma}{3}\hat{k}$$

The surface charge density on the top of slab 2 is equal to

$$\sigma_{top,2} = \overline{P}_2 \bullet \hat{n} = \left(-\frac{\sigma}{3}\hat{k}\right) \bullet \left(\hat{k}\right) = -\frac{\sigma}{3}$$

The surface charge density on the bottom of slab 2 is equal to

$$\sigma_{bottom,2} = \overline{P}_2 \bullet \hat{n} = \left(-\frac{\sigma}{3}\hat{k}\right) \bullet \left(-\hat{k}\right) = \frac{\sigma}{3}$$

f) The total charge above slab 1 is equal to σ - σ/2 = σ/2. This charge will produce an electric field in slab 1 equal to

$$\overline{E}_{above,1} = -\frac{\sigma}{4\varepsilon_0}\hat{k}$$

The total charge below slab 1 is equal to σ/2 - σ/3 + σ/3 - σ = - σ/2. This charge will produce an electric field in slab 1 equal to

$$\overline{E}_{below,1} = -\frac{\sigma}{4\varepsilon_0}\hat{k}$$

The total electric field in slab 1 is the vector sum of these two fields and is equal to

$$\overline{E}_1 = \overline{E}_{above,1} + \overline{E}_{below,1} = -\frac{\sigma}{2\varepsilon_0}\hat{k}$$

The total charge above slab 2 is equal to σ - σ/2 + σ/2 - σ/3 = 2σ/3. This charge will produce an electric field in slab 2 equal to

$$\overline{E}_{above,2} = -\frac{\sigma}{3\varepsilon_0}\hat{k}$$

The total charge below slab 1 is equal to σ/3 - σ = - 2σ/3. This charge will produce an electric field in slab 1 equal to

$$\overline{E}_{below,2} = -\frac{\sigma}{3\varepsilon_0}\hat{k}$$

The total electric field in slab 1 is the vector sum of these two fields and is equal to

$$\overline{E}_2 = \overline{E}_{above,2} + \overline{E}_{below,2} = -\frac{\sigma}{3\varepsilon_0}\hat{k} - \frac{\sigma}{3\varepsilon_0}\hat{k} = -\frac{2\sigma}{3\varepsilon_0}\hat{k}$$

These answers are in agreement with the results obtained in part b).

Example: Problem 4.20

A sphere of linear dielectric material has embedded in it a uniform free charge density ρ. Find the potential at the center of the sphere, if its radius is R and its dielectric constant is K.

The system has spherical symmetry and therefore the electric displacement $\overline{D}$ is easy to calculate since $\overline{\nabla} \bullet \overline{D} = \rho_{free}$ and $\overline{\nabla} \times \overline{D} = 0$. The calculation of $\overline{D}$ is very similar to the calculation of $\overline{E}$ using Gauss's law:

$$D(r) = \frac{1}{4\pi r^2} Q_{free,encl} = \frac{\rho R^3}{3r^2} \qquad r > R$$

$$D(r) = \frac{1}{4\pi r^2} Q_{free,encl} = \frac{1}{3}\rho r \qquad r < R$$

The corresponding electric field is equal to

$$E(r) = \frac{D(r)}{K\varepsilon_0} = \frac{\rho R^3}{3\varepsilon_0 r^2} \qquad r > R$$

$$E(r) = \frac{D(r)}{K\varepsilon_0} = \frac{1}{3K\varepsilon_0}\rho r \qquad r < R$$

Here we have used the fact that $K = 1$ in the region outside the sphere ($r > R$). The potential at the center of the sphere can be calculated using this electric field:

$$V(r) = -\int_\infty^0 \overline{E}(r) \bullet d\bar{l} = -\int_\infty^R \frac{\rho R^3}{3\varepsilon_0 r^2} dr - \int_R^0 \frac{1}{3K\varepsilon_0}\rho r dr =$$

$$= \left.\frac{\rho R^3}{3\varepsilon_0 r}\right|_\infty^R - \left.\frac{1}{6K\varepsilon_0}\rho r^2\right|_R^0 = \frac{\rho R^2}{3\varepsilon_0}\left\{1 + \frac{1}{2K}\right\}$$

The examples of calculations involving polarized material that have been discussed so far are either artificial, in the sense that the polarization is specified at the start, or highly symmetric, so that the electric displacement can be obtained directly from the free charge. In the next couple of examples we will encounter systems where these special conditions do not apply.

Example: Example 7 (Griffiths) and Problem 4.23

A sphere of linear dielectric material (dielectric constant *K*) is placed in an originally uniform electric field $\overline{E}_0$ (note: we will assume that this electric field is directed along the positive *z* axis).

a) Find the new field inside the sphere.

b) Solve for the field inside the sphere by the method of separation of variables. Note that: (1) *V* is continuous at *R*; (2) the discontinuity in the normal derivative of *V* at the surface is equal to $-\sigma_b / \varepsilon_0$; (3) because the dielectric is linear $\sigma_b = \varepsilon_0 \chi_e \left(\overline{E} \bullet \hat{n}\right) = \varepsilon_0 \chi_e \left(\overline{E} \bullet \hat{r}\right)$.

a) Suppose the electric field inside the sphere is equal to $\overline{E}_0$. Since the material is a linear dielectric the polarization is proportional to the total electric field:

$$\overline{P}_0 = \varepsilon_0 \chi_e \overline{E}_0$$

However, a uniformly polarized sphere with polarization $\overline{P}$ produces an internal electric field equal to

$$\overline{E} = -\frac{1}{3\varepsilon_0} \overline{P}$$

The electric field produced by the polarization of the sphere will therefore reduce the electric field inside the sphere by

$$\overline{E}_1 = -\frac{1}{3} \chi_e \overline{E}_0$$

This change in the electric field will change the polarization of the sphere by

$$\overline{P}_1 = \varepsilon_0 \chi_e \overline{E}_1 = -\frac{1}{3} \varepsilon_0 \chi_e{}^2 \overline{E}_0$$

This change in the polarization of the sphere will again change the electric field inside the sphere. This change of the electric field strength is equal to

$$\overline{E}_2 = -\frac{1}{3\varepsilon_0} \overline{P}_1 = \left(-\frac{\chi_e}{3}\right)^2 \overline{E}_0$$

This iterative process will continue indefinitely, and the final electric field will be equal to

$$\overline{E}_{final} = \overline{E}_0 + \overline{E}_1 + \overline{E}_2 + ... = \left(1 + \left(-\frac{\chi_e}{3}\right) + \left(-\frac{\chi_e}{3}\right)^2 + ...\right)\overline{E}_0 =$$

$$= \sum_{n=0}^{n}\left(-\frac{\chi_e}{3}\right)^n \overline{E}_0 = \frac{1}{1+\frac{\chi_e}{3}}\overline{E}_0 = \frac{3}{2+K}\overline{E}_0$$

The final polarization of the sphere is therefore equal to

$$\overline{P}_{final} = \varepsilon_0 \chi_e \overline{E}_{final} = \frac{3\varepsilon_0(K-1)}{2+K}\overline{E}_0$$

b) Since the dielectric will be uniformly polarized, all the bound charge will reside on the surface of the sphere:

$$\sigma_b = \varepsilon_0 \chi_e \left(\overline{E} \bullet \hat{n}\right) = \varepsilon_0 \chi_e \left(\overline{E} \bullet \hat{r}\right)$$

Therefore, the charge density is zero everywhere except on the surface of the sphere. The electrostatic potential of this system must therefore satisfy Laplace's equation (see Chapter 3). The most general solution of Laplace's equation for this system is

$$V(r,\theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta) \qquad r < R$$

$$V(r,\theta) = -E_0 r\cos\theta + \sum_{n=0}^{\infty} \frac{B_n}{r^{n+1}} P_n(\cos\theta) \qquad r > R$$

Note that the potential does not approach zero when r approaches infinity since the electric field at infinity is equal to $\overline{E}_0$. The electrostatic potential has to be continuous at $r = R$. Thus

$$\sum_{n=0}^{\infty} A_n R^n P_n(\cos\theta) = -E_0 R\cos\theta + \sum_{n=0}^{\infty} \frac{B_n}{R^{n+1}} P_n(\cos\theta)$$

This relation requires that

$$A_n R^n = \frac{B_n}{R^{n+1}} \qquad n \neq 1$$

$$A_1 R = -E_0 R + \frac{B_1}{R^2} \qquad n = 1$$

These two equations can be rewritten as

$$B_n = A_n R^{2n+1} \qquad n \neq 1$$

$$B_1 = A_1 R^3 + E_0 R^3 \qquad n = 1$$

The normal derivative of V at the surface of the sphere must satisfy the following boundary condition:

$$\left.\frac{\partial V}{\partial r}\right|_{r=R+} - \left.\frac{\partial V}{\partial r}\right|_{r=R-} = -\frac{\sigma_b}{\varepsilon_0}$$

Note that since the sphere is neutral, there is no free charge present. Therefore, the total surface charge on the sphere is equal to the bound surface charge. Substituting the general solution for V in this equation we obtain

$$-E_0 \cos\theta - \sum_{n=0}^{\infty}(n+1)\frac{B_n}{R^{n+2}} P_n(\cos\theta) - \sum_{n=0}^{\infty} n A_n R^{n-1} P_n(\cos\theta) = -\frac{\sigma_b}{\varepsilon_0}$$

This equation can be rewritten by using the expressions for B_n in terms of A_n with the following result:

$$\begin{aligned} -\frac{\sigma_b}{\varepsilon_0} &= -E_0 \cos\theta - \sum_{n=0}^{\infty}(n+1) A_n R^{n-1} P_n(\cos\theta) - 2E_0 \cos\theta - \sum_{n=0}^{\infty} n A_n R^{n-1} P_n(\cos\theta) = \\ &= -3E_0 \cos\theta - \sum_{n=0}^{\infty}(2n+1) A_n R^{n-1} P_n(\cos\theta) \end{aligned}$$

The bound charge is determined by the electric field, and therefore by the gradient of the potential:

$$\sigma_b = \varepsilon_0 \chi_e \left(\overline{E}_{r=R-} \bullet \hat{r}\right) = -\varepsilon_0 \chi_e \left.\frac{\partial V}{\partial r}\right|_{r=R-} = -\varepsilon_0 \chi_e \sum_{n=0}^{\infty} n A_n R^{n-1} P_n(\cos\theta)$$

Combining the last two equations we obtain

$$\chi_e \sum_{n=0}^{\infty} n A_n R^{n-1} P_n(\cos\theta) = -3E_0 \cos\theta - \sum_{n=0}^{\infty}(2n+1) A_n R^{n-1} P_n(\cos\theta)$$

This equation shows that for $n = 1$:

$$\chi_e A_1 = -3E_0 - 3A_1$$

and for $n \neq 1$:

$$n\chi_e A_n R^{n-1} = -(2n+1)A_n R^{n-1}$$

These two equations can be rewritten as

$$A_1 = -\frac{3E_0}{3+\chi_e}$$

and for $n \neq 1$

$$A_n = \frac{0}{\chi_e n + (2n+1)} = 0$$

The electrostatic potential since the sphere is thus equal to

$$V(r,\theta) = -\frac{3E_0}{3+\chi_e} r\cos\theta = -\frac{3E_0}{2+K} z$$

We conclude that the electrostatic potential inside the sphere only depends on the z coordinate. The electric field inside the sphere can be obtained from the gradient of the electrostatic potential:

$$\overline{E} = -\overline{\nabla} V = -\frac{\partial V}{\partial z}\hat{k} = \frac{3E_0}{2+K}\hat{k}$$

which is identical to the result we obtained in a).

Example: Problem 4.35

Prove the following uniqueness theorem: A region S contains a specified free charge distribution ρ_f and various pieces of linear dielectric material, with the susceptibility of each one given. If the potential is specified on the boundary of S (and $V = 0$ at infinity) then the potential throughout S is uniquely defined.

Suppose that there are two different solutions V_1 and V_2. The corresponding electric fields are $\overline{E}_1 = -\overline{\nabla} V_1$ and $\overline{E}_2 = -\overline{\nabla} V_2$, respectively. The corresponding electric displacements are $\overline{D}_1 = \varepsilon\overline{E}_1$ and $\overline{D}_2 = \varepsilon\overline{E}_2$. Consider a third function $V_3 = V_2 - V_1$. Since V_1 and V_2 must have the same value on the border, $V_3 = 0$ there. Now consider the volume integral (over volume S) of $\overline{\nabla} \bullet \left(V_3 \overline{D}_3 \right)$:

$$\int_{Volume\ S} \overline{\nabla} \bullet \left(V_3 \overline{D}_3 \right) d\tau = \int_{Surface\ S} V_3 \overline{D}_3 \bullet d\overline{a} = 0$$

since $V_3 = 0$ on the surface of volume S. The left-hand side of this equation can be rewritten as

$$\int_{Volume\ S} \overline{\nabla} \bullet \left(V_3 \overline{D}_3 \right) d\tau = \int_{Volume\ S} \left(\overline{\nabla} V_3 \bullet \overline{D}_3 \right) d\tau + \int_{Volume\ S} V_3 \left(\overline{\nabla} \bullet \overline{D}_3 \right) d\tau = 0$$

But the divergence of $\overline{D}_3$ is equal to zero since

$$\left(\overline{\nabla} \bullet \overline{D}_3 \right) = \left(\overline{\nabla} \bullet \overline{D}_2 \right) - \left(\overline{\nabla} \bullet \overline{D}_1 \right) = \rho_{free} - \rho_{free} = 0$$

Therefore

$$\int_{Volume\ S} \left(\overline{\nabla} V_3 \bullet \overline{D}_3 \right) d\tau + \int_{Volume\ S} V_3 \left(\overline{\nabla} \bullet \overline{D}_3 \right) d\tau = \int_{Volume\ S} \left(\overline{\nabla} V_3 \bullet \overline{D}_3 \right) d\tau = 0$$

The integral on the right-hand side of this equation can be rewritten in terms of $\overline{E}_3$ using the following relations:

$$\overline{\nabla} V_3 = \overline{\nabla} V_2 - \overline{\nabla} V_1 = -\overline{E}_2 + \overline{E}_1 = -\overline{E}_3$$

and

$$\overline{D}_3 = \overline{D}_2 - \overline{D}_1 = \varepsilon \overline{E}_2 - \varepsilon \overline{E}_1 = \varepsilon \overline{E}_3$$

Therefore,

$$\int_{Volume\ S} \left(\overline{\nabla} V_3 \bullet \overline{D}_3 \right) d\tau = \int_{Volume\ S} \left(\left(-\overline{E}_3 \right) \bullet \left(\varepsilon \overline{E}_3 \right) \right) d\tau = -\varepsilon \int_{Volume\ S} E_3^{\ 2} d\tau = 0$$

Since $\varepsilon > 0$ this equation can only be satisfied if $E_3 = 0$. This requires that

$$\overline{E}_1 = \overline{E}_2$$

and

$$V_1 = V_2$$

everywhere. We therefore conclude that there are no two different electrostatic potentials that satisfy the same boundary conditions. The electrostatic potential is therefore uniquely defined if its value is specified on the surface of the volume S.

Example: Problem 4.36

A conducting sphere at potential V_0 is half embedded in linear dielectric material of susceptibility χ_e, which occupies the region $z < 0$ (see Figure 4.11). *Claim*: the potential everywhere is exactly the same as it would have been in the absence of the dielectric! Check this claim as follows:

a) Write down the formula for the suggested potential $V(r)$, in terms of V_0, R, and r. Use it to determine the field, the polarization, the bound charge, and the free charge distribution on the sphere.
b) Show that the total charge configuration would indeed produce the potential $V(r)$.
c) Appeal to the uniqueness theorem in Problem 4.35 to complete the argument.

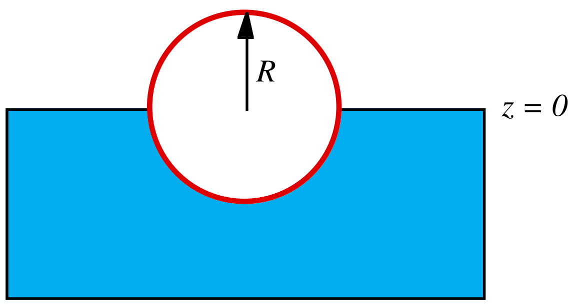

Figure 4.11. Problem 4.36.

a) In the absence of the dielectric, the electrostatic potential of this system is constant inside the sphere and is given by

$$V(r) = V_0 \frac{R}{r}$$

in the region outside the sphere. The electric field in the region outside the sphere is equal to the gradient of V and is therefore given by

$$\overline{E}(r) = -\overline{\nabla} V(r) = V_0 \frac{R}{r^2} \hat{r}$$

If this solution satisfies the boundary conditions on the surface of the sphere when the dielectric is present then it is the only solution (uniqueness theorem of problem 4.35). The boundary conditions for the electrostatic potential are:

1. V is continuous on the surface of the sphere. This boundary condition is satisfied by the proposed solution.

2. The difference in the normal derivative of V on the surface of the sphere is equal to

$$\left.\frac{\partial V}{\partial r}\right|_{r=R+} - \left.\frac{\partial V}{\partial r}\right|_{r=R-} = -\frac{\sigma_{total}}{\varepsilon_0}$$

   For the proposed solution this requires that

$$\sigma_{total} = -\varepsilon_0\left(\left.\frac{\partial V}{\partial r}\right|_{r=R+} - \left.\frac{\partial V}{\partial r}\right|_{r=R-}\right) = \frac{\varepsilon_0 V_0}{R}$$

   which shows that the total charge is uniformly distributed across the surface of the sphere.

The polarization of the dielectric material in the region $z < 0$ (and $r > R$) can be obtained from the electric field:

$$\overline{P} = \varepsilon_0 \chi_e \overline{E} = \varepsilon_0 \chi_e V_0 \frac{R}{r^2}\hat{r}$$

In the region $z > 0$ the polarization is equal to zero since no dielectric material is present there. The bound surface charge on the surface of the dielectric is equal to

$$\sigma_b = \overline{P} \bullet \hat{n}$$

where $\hat{n}$ is the surface vector (perpendicular to the surface and pointing out of the dielectric). For the spherical surface $\hat{n} = -\hat{r}$ and thus

$$\sigma_b = \overline{P}(R) \bullet \hat{n} = -\varepsilon_0 \chi_e V_0 \frac{R}{R^2}(\hat{r} \bullet \hat{r}) = -\varepsilon_0 \chi_e V_0 \frac{1}{R}$$

There is no bound charge on the flat surface ($z = 0$) of the dielectric since $\hat{n} \perp \hat{r}$ is there. The bound volume charge is zero everywhere since

$$\rho_b = -\overline{\nabla} \bullet \overline{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \varepsilon_0 \chi_e V_0 \frac{R}{r^2}\right) = 0$$

The free charge on the surface of the sphere can be determined from the electric displacement $\overline{D}$. The electric displacement $\overline{D}$ can be obtained from the electric field. In the region above the dielectric ($z > 0$) and outside the sphere ($r > R$) the electric displacement $\overline{D}$ is equal to

$$\overline{D}(\overline{r}) = \varepsilon_0 \overline{E}(\overline{r}) + \overline{P}(\overline{r}) = \varepsilon_0 \chi_e V_0 \frac{R}{r^2}\hat{r}$$

In the region $z < 0$ and outside the sphere ($r > R$) the electric displacement $\overline{D}$ is equal to

$$\overline{D}(\overline{r}) = \varepsilon_0 \overline{E}(\overline{r}) + \overline{P}(\overline{r}) = \varepsilon_0 (1 + \chi_e) V_0 \frac{R}{r^2}\hat{r}$$

The free charge on the bottom hemisphere and part of the $z = 0$ plane (see Figure 4.12a) is equal to

$$Q_{free} = \oint_{Bottom\ Surface} \overline{D}(\overline{r}) \bullet d\overline{a} = 2\pi R^2 \varepsilon_0 (1 + \chi_e) V_0 \frac{1}{R}$$

There is no contribution to the surface charge from the $z = 0$ plane since $\overline{D}(\overline{r}) \perp d\overline{a}$ there. The free charge density on the bottom hemisphere is therefore equal to

$$Q_{free,z<0} = \frac{Q_{free}}{2\pi R^2} = \varepsilon_0 (1 + \chi_e) V_0 \frac{1}{R}$$

In the same manner we can calculate the free charge density on the top hemisphere:

$$Q_{free,z>0} = \frac{Q_{free}}{2\pi R^2} = \varepsilon_0 V_0 \frac{1}{R}$$

The total charge density (bound charge + free charge) on the surface is therefore equal to

$$\sigma_{total,z<0} = \sigma_{free,z<0} + \sigma_{bound,z<0} = \varepsilon_0 (1 + \chi_e) V_0 \frac{1}{R} - \varepsilon_0 \chi_e V_0 \frac{1}{R} = \varepsilon_0 V_0 \frac{1}{R}$$

$$\sigma_{total,z>0} = \sigma_{free,z>0} + \sigma_{bound,z>0} = \varepsilon_0 V_0 \frac{1}{R} + 0 = \varepsilon_0 V_0 \frac{1}{R}$$

Therefore, the total charge on the surface of the sphere is distributed uniformly, and has a value consistent with the boundary condition for the normal derivative of V. Since the proposed solution satisfies the boundary conditions for V it will be the only correct solution.

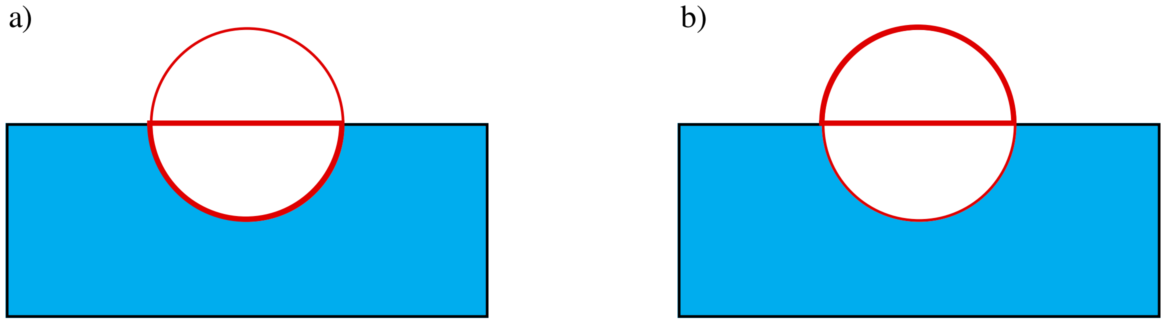

Figure 4.12. Determination of free charge in Problem 4.36.

4.5. Energy in dielectric systems

Consider a capacitor with capacitance C and charged up to a potential V. The total energy stored in the capacitor is equal to the work done during the charging process:

$$W = \frac{1}{2} C V^2$$

If the capacitor is filled with a linear dielectric (dielectric constant K) than the total capacitance will increase by a factor K:

$$C = K C_{vac}$$

and consequently the energy stored in the capacitor (when held at a constant potential) is increased by a factor K. A general expression for the energy of a capacitor with dielectric materials present can be found by studying the charging process in detail. Consider a free charge ρ_{free} held at a potential V. During the charging process the free charge is increased by $\Delta\rho_{free}$. The work done on the extra free charge is equal to

$$\Delta W = \int_{Volume} \Delta\rho_{free} V d\tau$$

Since the divergence of the electric displacement $\overline{D}$ is equal to the free charge density ρ_{free}, the divergence of $\Delta\overline{D}$ is equal to $\Delta\rho_{free}$. Therefore,

$$\Delta W = \int_{Volume} \left[\overline{\nabla} \bullet \Delta\overline{D}\right] V d\tau$$

Using the following relation

$$\overline{\nabla} \bullet (V\Delta\overline{D}) = [\overline{\nabla} \bullet \Delta\overline{D}]V + (\overline{\nabla}V) \bullet (\Delta\overline{D})$$

we can rewrite the expression for ΔW as

$$\Delta W = \int_{Volume} [\overline{\nabla} \bullet (V\Delta\overline{D})]d\tau - \int_{Volume} [(\overline{\nabla}V) \bullet (\Delta\overline{D})]d\tau$$

The first term on the right-hand side of this equation can be rewritten as

$$\int_{Volume} [\overline{\nabla} \bullet (V\Delta\overline{D})]d\tau = \int_{Surface} (V\Delta\overline{D}) \bullet d\overline{a} = 0$$

since the product of potential and electric displacement approach zero faster than $1/r^2$ when r approached infinity. Therefore,

$$\Delta W = -\int_{Volume} [(\overline{\nabla}V) \bullet (\Delta\overline{D})]d\tau = \int_{Volume} [\overline{E} \bullet (\Delta\overline{D})]d\tau$$

Assuming that the materials present in the system are linear dielectrics then

$$\overline{D} = \varepsilon\overline{E}$$

This relation can be used to rewrite $\overline{E} \bullet (\Delta\overline{D})$:

$$\overline{E} \bullet (\Delta\overline{D}) = \overline{E} \bullet (\varepsilon\Delta\overline{E}) = \frac{1}{2}\Delta(\varepsilon\overline{E} \bullet \overline{E}) = \frac{1}{2}\Delta(\overline{D} \bullet \overline{E})$$

The expression for ΔW can thus be rewritten as

$$\Delta W = \frac{1}{2}\Delta \int_{Volume} (\overline{D} \bullet \overline{E})d\tau$$

The total work done during the charging process is therefore equal to

$$W = \frac{1}{2}\int_{Volume} (\overline{D} \bullet \overline{E})d\tau$$

Note: this equation can be used to calculate the energy for a system that contains linear dielectrics. If some materials in the system are non-linear dielectrics than the derivation given above is not correct ($\overline{E} \bullet (\Delta\overline{D}) \neq 0.5\Delta(\overline{D} \bullet \overline{E})$ for non-linear dielectrics).

Example: Problem 4.26

A spherical conductor, of radius a, carries a charge Q. It is surrounded by linear dielectric material of susceptibility χ_e, out to a radius b. Find the energy of this configuration.

Since the system has spherical symmetry the electric displacement $\overline{D}$ is completely determined by the free charge. It is equal to

$$\overline{D}(\bar{r}) = \frac{1}{4\pi r^2} Q_{encl} = 0 \qquad r < a$$

$$\overline{D}(\bar{r}) = \frac{1}{4\pi r^2} Q_{encl} = \frac{1}{4\pi} \frac{Q}{r^2} \qquad r > a$$

Since we are dealing with linear dielectrics, the electric field $\overline{E}$ is equal to $\overline{D} / (\varepsilon_0 (1 + \chi_e))$. Taking into account that the susceptibility of vacuum is zero and the susceptibility of a conductor is infinite we obtain for $\overline{E}$:

$$\overline{E}(\bar{r}) = 0 \qquad r < a$$

$$\overline{E}(\bar{r}) = \frac{\overline{D}(\bar{r})}{\varepsilon_0 (1 + \chi_e)} = \frac{1}{4\pi\varepsilon_0 (1 + \chi_e)} \frac{Q}{r^2} \qquad a < r < b$$

$$\overline{E}(\bar{r}) = \frac{\overline{D}(\bar{r})}{\varepsilon_0} = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} \qquad b < r$$

The scalar product $\overline{D} \bullet \overline{E}$ is equal to $|\overline{D}||\overline{E}|$ since $\overline{E}$ and $\overline{D}$ are parallel, everywhere. The energy of the system is equal to

$$\begin{aligned} W &= \frac{1}{2} \int_{Volume} (\overline{D} \bullet \overline{E}) d\tau = 2\pi \int_a^\infty |\overline{D}| \, |\overline{E}| \, r^2 dr = \\ &= 2\pi \int_a^b \frac{1}{16\pi^2 \varepsilon_0 (1 + \chi_e)} \frac{Q^2}{r^4} r^2 dr + 2\pi \int_b^\infty \frac{1}{16\pi^2 \varepsilon_0} \frac{Q^2}{r^4} r^2 dr = \\ &= \frac{Q^2}{8\pi\varepsilon_0} \left\{ \frac{1}{(1 + \chi_e)} \left(\frac{1}{a} - \frac{1}{b} \right) + \frac{1}{b} \right\} = \frac{Q^2}{8\pi\varepsilon_0 (1 + \chi_e)} \left\{ \frac{1}{a} + \frac{\chi_e}{b} \right\} \end{aligned}$$

4.6. Forces on dielectrics

A dielectric slab placed partly between the plates of a parallel-plate capacitor will be pulled inside the capacitor. This force is a result of the fringing fields around the edges of the parallel-plate capacitor (see Figure 4.13). **Note: the field outside the capacitor can not be zero** since otherwise the line integral of the electric field around a closed loop, partly inside the capacitor and partly outside the capacitor, would not be equal to zero.

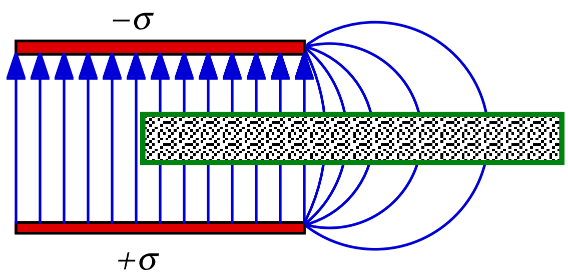

Figure 4.13. Fringing fields.

Inside the capacitor the electric field is uniform. The electric force exerted by the field on the positive bound charge of the dielectric is directed upwards and is canceled by the electric force on the negative bound charge (see Figure 4.14). Outside the capacitor the electric field is not uniform and the electric force acting on the positive bound charge will not be canceled by the electric force acting on the negative bound charge. For the system shown in Figure 4.14 the vertical components of the two forces (outside the capacitor) will cancel, but the horizontal components are pointing in the same direction and therefore do not cancel. The result is a net force acting on the slab, directed towards the center of the capacitor.

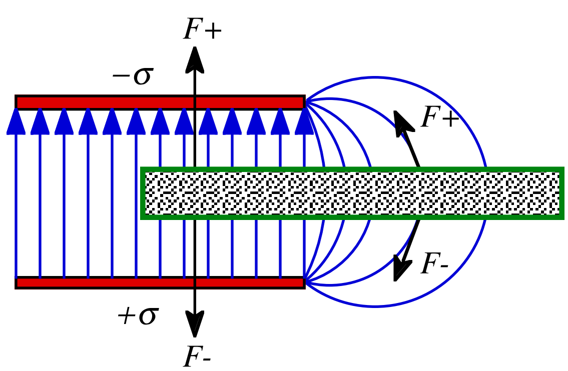

Figure 4.14. Forces on dielectric.

A direct calculation of this force requires a knowledge of the fringing fields of the capacitor which are often not well known and difficult to calculate. An alternative method that can be used is to determine this force is to calculate the change in the energy of the system when the dielectric is displaced by a distance ds. The work to be done to pull the dielectric out by an infinitesimal distance ds is equal to

$$dW = \overline{F}_{us} ds$$

where $\overline{F}_{us}$ is the force provided by us to pull the slab out of the capacitor. This force must just be equal in magnitude but directed in a direction opposite to the force $\overline{F}_{field}$ exerted by the electric field on the slab. Thus

$$\overline{F}_{field} = -\overline{F}_{us} = -\frac{dW}{ds}$$

The work done by us to move the slab must be equal to the change in the energy of the capacitor (conservation of energy). Consider the situation shown in Figure 4.15 where the slab of dielectric is inserted to a depth s in the capacitor. The capacitance of this system is equal to

$$C = C_{vac} + C_{diel} = \frac{\varepsilon_0 (w - s) a}{d} + K\frac{\varepsilon_0 s a}{d} = \frac{\varepsilon_0 a}{d}(w + \chi_e s)$$

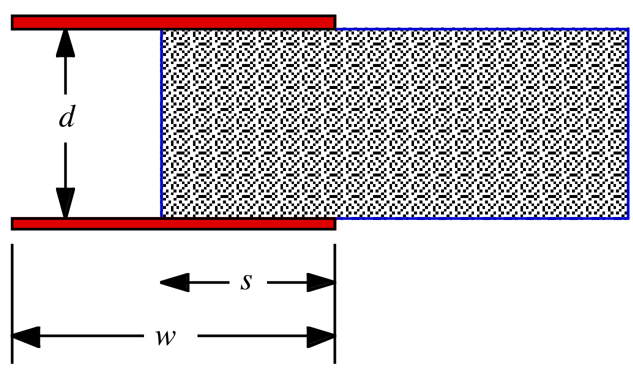

Figure 4.15. Calculation of $\overline{F}_{field}$.

If the total charge on the top plate is Q then the energy stored in the capacitor is equal to

$$W = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}Q^2 \frac{d}{\varepsilon_0 a (w + \chi_e s)}$$

The force on the dielectric can now be calculated and is equal to

$$F_{field} = -\frac{dW}{ds} = -\frac{1}{2}\frac{Q^2}{C^2}\frac{dC}{ds}$$

Example: Problem 4.28

Two long coaxial cylindrical metal tubes (inner radius a, outer radius b) stand vertically in a tank of dielectric oil (susceptibility χ_e, mass density ρ). The inner one is maintained at potential V, and the outer one is grounded. To what height h does the oil rise in the space between the tubes?

The height of the oil is such that the electric force on the oil balances the gravitational force. The capacitance of an empty cylindrical capacitor of height H is equal to

$$C = \frac{2\pi\varepsilon_0 H}{\ln\left(\frac{b}{a}\right)}$$

If the oil rises to a height h then the capacitance of the capacitor is equal to

$$C_{tot} = C_{vac} + C_{oil} = \frac{2\pi\varepsilon_0 (H-h)}{\ln\left(\frac{b}{a}\right)} + \left(1+\chi_e\right)\frac{2\pi\varepsilon_0 h}{\ln\left(\frac{b}{a}\right)} = \frac{2\pi\varepsilon_0}{\ln\left(\frac{b}{a}\right)}\left(H + \chi_e h\right)$$

The electric force on the dielectric (the oil) is equal to

$$F_{el} = \frac{1}{2}V^2\frac{dC}{dh} = \chi_e \frac{\pi\varepsilon_0}{\ln\left(\frac{b}{a}\right)}V^2$$

and is directed upwards The gravitational force acting on the oil is equal to

$$F_{grav} = \pi\left(b^2 - a^2\right)h\rho g$$

and is directed downwards. In the equilibrium position $\left|F_{grav}\right| = \left|F_{el}\right|$. Thus

$$\pi\left(b^2 - a^2\right)h\rho g = \chi_e \frac{\pi\varepsilon_0}{\ln\left(\frac{b}{a}\right)}V^2$$

or

$$h = \chi_e \frac{\varepsilon_0}{\rho g\left(b^2 - a^2\right)\ln\left(\frac{b}{a}\right)} V^2$$

For a linear dielectric, the polarization $\overline{P}$ is proportional to the total macroscopic field $\overline{E}_{total}$:

$$\overline{P} = \varepsilon_0 \chi_e \overline{E}_{total}$$

The polarization of the dielectric is equal to the vector sum of the polarization $\overline{p}$ of the individual atoms or molecules:

$$\overline{P} = N\,\overline{p}$$

where N is the number of atoms or molecules per unit volume. The polarization of an individual atom or molecule is proportional to the microscopic field at the position of the atom or molecule due to everything except the particular atom or molecule under consideration:

$$\overline{p} = \alpha\,\overline{E}_{else}$$

The dipole moment of the atom or molecule will generate an electric field at its center equal to

$$\overline{E}_{self} = -\frac{1}{4\pi\varepsilon_0}\frac{\overline{p}}{R^3}$$

where R is the radius of the atom or molecule. The total macroscopic field seen by the atom or molecule is there for equal to

$$\overline{E}_{total} = \overline{E}_{self} + \overline{E}_{else} = -\frac{1}{4\pi\varepsilon_0}\frac{\alpha\overline{E}_{else}}{R^3} + \overline{E}_{else} = \left(1 - \frac{\alpha}{4\pi\varepsilon_0 R^3}\right)\overline{E}_{else} = \left(1 - \frac{N\alpha}{3\varepsilon_0}\right)\overline{E}_{else}$$

where N is the number of atoms per unit volume. The total polarization of the dielectric is thus equal to

$$\overline{P} = N\,\overline{p} = N\alpha\,\overline{E}_{else} = \frac{N\alpha}{\left(1 - \frac{N\alpha}{3\varepsilon_0}\right)}\overline{E}_{total} = \varepsilon_0 \chi_e \overline{E}_{total}$$

Therefore

$$\chi_e = \frac{\frac{N\alpha}{\varepsilon_0}}{1 - \frac{N\alpha}{3\varepsilon_0}} = \frac{3N\alpha}{3\varepsilon_0 - N\alpha}$$

This equation can be rewritten in terms of the dielectric constants *K* as

$$K - 1 = \frac{3N\alpha}{3\varepsilon_0 - N\alpha}$$

or

$$\alpha = \frac{3\varepsilon_0}{N} \frac{K-1}{K+2}$$

This equation shows that a measurement of the macroscopic parameter *K* can be used to obtain information about the microscopic parameter α. This equation is known as the **Clausius-Mossotti formula** or the **Lorentz-Lorenz equation**.