SEQUENCE OF FUNCTIONS

1. Introduction

In this chapter we shall deal with sequences $\{f_n\}$ whose terms are real-valued functions having a common domain on the real line $\mathbb{R}$.

Definition1.1. Suppose $\{f_n\}$ is a sequence of functions defined on $S \subseteq \mathbb{R}$ and suppose that, for each x in S the real sequence $\{f_n(x)\}$ (whose terms are corresponding function values) converges.Then the function $f: \ S \to \mathbb{R}$ defined by

$$f(x) = \lim_{n\to\infty} f_n(x) \ , \forall x \in S$$

f is called the limit function or pointwise limit of the sequence $\{f_n\}$ and we say $\{f_n\}$ converges to f on S or $\{f_n\}$ converges pointwise to f on S. In this case $\{f_n\}$ is said to be a convergent sequence of functions on S.

The chief interest in this chapter is to determine whether or not some important properties of a convergent sequence of functions are inherited by its limit function.

To be explicit , let us consider a convergent sequence of function on some $S \subseteq \mathbb{R}$ with f as its limit function on S . we would like to deal with the following type of questions :

(i) if all the functions in $\{f_n\}$ are continuous at some fixed c in S then is it true that f is also continuous at c ?

(ii) if all the functions in $\{f_n\}$ are bounded on S then is it true that f is also bounded on S ?

(iii) if $S = [a, b]$ and all the function $\{f_n\}$ are integrable on $[a, b]$ then is it true that f is also integrable on $[a, b]$? . In case f is integrable what is the relation between the sequence of integrals $\{\int_a^b f_n(x)dx\}$ and $\int_a^b f(x)dx$?

(iv)if $S = [a, b]$ and all the function in$\{f_n\}$ are differentiable at some fixed c in $[a, b]$ then is it true that f is also differentiable at c ? In case f is differentiable at c what is the relation between $\{f_n'(c)\}$ and $f'(c)$?

Before trying to answer these questions let us have a few examples of sequence of functions converging pointwise on some subsets of $\mathbb{R}$.

Examples of sequences of functions :

The following examples illustrate some of the possibilities that might arise when we form the limit function of a sequence of functions:

Example 1: For each $n \in \mathbb{N}$, suppose that $f_n : [0,1] \to \mathbb{R}$ is given by $f_n(x) = x^n, \forall x \in [0,1]$. Then the limit function f of the sequence of functions $\{f_n\}$ on $[0,1]$ is given by

$$f(x) = \begin{cases} 0 & \text{if } x \in [0,1) \\ 1 & \text{if } x = 1 \end{cases}$$

Example 2: For each $n \in \mathbb{N}$, suppose that $f_n : \mathbb{R} \to \mathbb{R}$ is given by $f_n(x) = \frac{1}{1+n^2x^2}, \forall x \in \mathbb{R}$. Then the limit function f of the sequence of functions $\{f_n\}$ on $\mathbb{R}$ is given by

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R}-\{0\} \\ 1 & \text{if } x = 0 \end{cases}$$

Example 3: For each $n \in \mathbb{N}$, suppose that $f_n : \mathbb{R} \to \mathbb{R}$ is given by $f_n(x) = \frac{x^{2n}}{1+x^{2n}}, \forall x \in \mathbb{R}$. Then the limit function f of the sequence of functions $\{f_n\}$ on $\mathbb{R}$ is given by

$$f(x) = \begin{cases} 0 & \text{if } |x| < 1 \\ \frac{1}{2} & \text{if } |x| = 1 \\ 1 & \text{if } |x| > 1 \end{cases}$$

Example 4: For each $n \in \mathbb{N}$, suppose that $f_n : [0,1] \to \mathbb{R}$ is given by

$$f_n(x) = \begin{cases} 1 - nx^2 & \text{if } x \in [0, \frac{1}{\sqrt{n}}] \\ 0 & \text{if } x \in (\frac{1}{\sqrt{n}}, 1] \end{cases}$$

Then the limit function f of the sequence of functions $\{f_n\}$ on $[0,1]$ is given by

$$f(x) = \begin{cases} 0 & \text{if } x \in (0,1] \\ 1 & \text{if } x = 0 \end{cases}$$

Example 5: For each $n \in \mathbb{N}$, suppose that $f_n : (0,1) \to \mathbb{R}$ is given by $f_n(x) = 1 + x + x^2 + \cdots + x^{n-1}, \forall x \in (0,1)$.
Then the limit function f of the sequence of functions $\{f_n\}$ on $(0,1)$ is given by $f(x)= \frac{1}{1-x}, \forall x \in (0,1)$.

Example 6: Let $r_1, r_2, \cdots$ be an enumeration of the rationals in $[0,1]$.Now, for each $n \in \mathbb{N}$, let us define $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 1 & \text{if } x \in \{r_1, r_2, \cdots, r_n\} \\ 0 & \text{if } x \in [0,1] - \{r_1, r_2, \cdots, r_n\} \end{cases}$$

Then the limit function f of the sequence of functions $\{f_n\}$ on $[0,1]$ is given by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational in [0,1]} \\ 0 & \text{if } x \text{ is irrational in [0,1]} \end{cases}$$

Example 7: For each $n \in \mathbb{N}$, suppose that $f_n : [0,1] \to \mathbb{R}$ is given by $f_n(x) = n^2x(1-x^2)^n, \forall x \in [0,1]$.
Then the limit function f of the sequence of functions $\{f_n\}$ on $[0,1]$ is given by $f(x) = 0, \forall x \in$ [0,1] .

Example 8: For each $n \in \mathbb{N}$, suppose that $f_n : [0,1] \to \mathbb{R}$ is given by $f_n(x) = nx(1-x^2)^n, \forall x \in [0,1]$.
Then the limit function f of the sequence of functions $\{f_n\}$ on $[0,1]$ is given by $f(x) = 0, \forall x \in$ [0,1] .

Example 9: For each $n \in \mathbb{N}$, suppose that $f_n : [0,1] \to \mathbb{R}$ is given by $f_n(x) = \frac{\sin nx}{1+nx}, \forall x \in [0,1]$.
Then the limit function f of the sequence of functions $\{f_n\}$ on $[0,1]$ is given by $f(x) = 0, \forall x \in$ [0,1] .

Example 10: For each $n \in \mathbb{N}$, suppose that $f_n : \mathbb{R} \to \mathbb{R}$ is given by $f_n(x) = x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \cdots + \frac{x^2}{(1+x^2)^n}, \forall x \in \mathbb{R}$.
Then the limit function f of the sequence of functions $\{f_n\}$ on $\mathbb{R}$ is given by

$$f(x) = \begin{cases} 1 + x^2 & \text{if } x \in \mathbb{R} - \{0\} \\ 0 & \text{if } x = 0 \end{cases}$$

Observations:

Note1 : In example1, each f_n is continuous at $x = 1$ but f is not continuous at $x = 1$. So continuity at a point may not be preserved under pointwise convergence. Similar remarks can also be made from the examples 2,3,4,6,10. However in examples 5,7,8,9 continuity is preserved everywhere.

Note2 : Note that in example 6, each f_n is discontinuous at n points only but the limit function becomes totally discontinuous. Clearly for each n, f_n is Riemann integrable on $[0,1]$ but the limit function is not Riemann integrable on $[0,1]$. So pointwise limit of a sequence of Riemann integrable function need not be Riemann integrable.

Note3 : In example 5, each f_n is bounded on $(0,1)$ [Since any polynomial function on any bouded interval is bounded] but the limit function f is unbounded on $(0,1)$. So, pointwise convergence fails to preserve even boundedness on a set.

Note4 : In each of example 7 and 8, each f_n is a continuous function on $[0,1]$ and the limit function f is also continuous on $[0,1]$. So in either example, under the pointwise convergence each of continuity at any point of $[0,1]$, boundedness on $[0,1]$ and Riemann integrability on $[0,1]$ are preserved. However in Ex.7,
$\int_0^1 f_n(x)dx = \frac{n^2}{2(n+1)} \to \infty$ as $n \to \infty$ and in Ex.8, $\int_0^1 f_n(x)dx = \frac{n}{2(n+1)} \to \frac{1}{2}$ as $n \to \infty$ whereas in either example, $\int_0^1 f(x)dx = 0$. So if $\{f_n\}$ is a sequence of continuous functions on $[0,1]$ converging pointwise to a function f, the equality

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx$$

may not be true; in fact, the limit of the integral need not be equal to the integral of the limit, even if both are finite.

Note5 : In example 9, $\{f_n\}$ is a sequence of continuous functions with continuous limit function f on $[0,1]$ and

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx$$

.

We now state the formal definition of pointwise convergence of a sequence of function $\{f_n\}$.

Definition: Let $\{f_n\}$ be a sequence of function defined on S,where $S \subset \mathbb{R}$. We say that $\{f_n\}$ converges pointwise to a function f on S if $\forall x \in S$ and $\forall \epsilon > 0$ there is a positive integer N such that $|f_n(x) - f(x)| < \epsilon \ \forall n \geq N$.

In the above definition it is very important to note that N depends not only on ϵ but also on x.To see this let us look at the following example.

Consider Ex.1. Fix $\epsilon = \frac{1}{2}$.Then for each $x \in [0,1]$, $\exists N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \frac{1}{2} \ \forall n \geq N$. If $x = 0$ or $x = 1$, then the above relation is true for $N = 1$. However if we take $x = \frac{3}{4}$, then it is easy to see that the above relation is true for $N = 3$.Similarly if we take $x = \frac{9}{10}$, then $N = 7$. This shows that N depends on the points x. We now examine whether there is a positive integer N which is independent on x.Suppose that there is a positive integer N such that the above relation is true for all x. That means $x^n < \frac{1}{2} \ \forall n \geq N$ and $\forall x \in [0,1)$.This implies that $x^N < \frac{1}{2}$ $(0 \leq x < 1)$. Taking limit as $x \to 1^-$, we get $1 \leq \frac{1}{2}$ which is a contradiction. Thus no such N exist in this case.
We have seen that the answers to each questions (i),(ii) and (iii) posed earlier are negative. Thus we see that pointwise convergence is usually not strong enough to transfer any of the properties viz. continuity, boundedness, Riemann Integrability from the individual terms f_n to the limit function f. Now we shall study a stronger type of convergence, called Uniform Convergence, that preserves

these properties.

Definition (Uniform Convergence): A sequence of functions $\{f_n\}$ converges uniformly to a function f on S if $\forall \epsilon > 0$ there is a positive integer N such that $|f_n(x) - f(x)| < \epsilon \ \forall n \geq N$ and $\forall x \in S$.

In case $\{f_n\}$converges uniformly to f on S we say that f is the uniform limit of the sequence of functions $\{f_n\}$ and $\{f_n\}$is said to be uniformly convergent sequence of functions on S.
Obviously every uniformly convergent sequence of functions is pointwise convergent and uniform limit of any uniformly convergent sequence of functions on $S \subset \mathbb{R}$ must be the pointwise limit of the same sequence of functions on S.
Although we shall establish that uniform convergence preserves continuity,Boundedness and integrability, it will also be shown that uniform convergence is not sufficient to preserve differentiability. So, even under such stronger type of convergence answer to the question analogous to (iv) posed earlier will remain negative.
Before dealing with these theoretical aspects let us look at few examples of convergent sequences of functions on some subsets of $\mathbb{R}$ and test whether the convergence is uniform on the common domain of the corresponding sequence of functions or not.

Example 11: For each $n \in \mathbb{N}$, suppose that $f_n : \mathbb{R} \to \mathbb{R}$ is defined by $f_n(x) = \frac{Sinnx}{n} \ \forall x \in \mathbb{R}$.
Then the limit function f of this sequence of functions $\{f_n\}$ is given by $f(x) = 0 \ \forall x \in \mathbb{R}$, for keeping x fixed at any point of $\mathbb{R}$, $0 \leq |f_n(x)| \leq \frac{1}{n} \to 0$ as $n \to \infty$.
Now to test whether $\{f_n\}$ converges uniformly to f on $\mathbb{R}$ or not :
Let $\epsilon > 0$ be a given real.
Note that $\forall x \in \mathbb{R}$, $|f_n(x) - f(x)| = |\frac{Sinnx}{n} - 0| \leq \frac{1}{n} \ \forall n$. Let us choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$.
Then $\forall x \in \mathbb{R}$, $\forall n \geq N$ we have $|f_n(x) - f(x)| < \epsilon$.
Thus the sequence $\{f_n\}$ converges uniformly to f on $\mathbb{R}$.
Example 12: For each $n \in \mathbb{N}$, suppose that $f_n : \mathbb{R} \to \mathbb{R}$ is defined by $f_n(x) = \frac{Sinnx}{\sqrt{n}} \ \forall x \in \mathbb{R}$.
Then as Ex.11 one can prove that the limit function f of $\{f_n\}$ on $\mathbb{R}$ is given by $f(x) = 0 \ \forall x \in \mathbb{R}$ and moreover the sequence $\{f_n\}$ converges uniformly to f on $\mathbb{R}$.
Example 13: For each $n \in \mathbb{N}$, suppose that $g_n : (0,1) \to \mathbb{R}$ is defined by $g_n(x) = \frac{x}{nx+1} \ \forall x \in (0,1)$.
Then the limit function g of the sequence of functions $\{g_n\}$ on $(0,1)$ is given by $g(x) = 0 \ \forall x \in (0,1)$.
Now to test whether $\{g_n\}$ converges uniformly to g on $(0,1)$ or not :
Let $\epsilon > 0$ be a given real.
Note that $\forall x \in (0,1)$, $|g_n(x) - g(x)| = \frac{x}{nx+1} < \epsilon$ if $\frac{1}{\epsilon} - \frac{1}{x} < n$. Let us choose $N \in \mathbb{N}$ such that

$N > \frac{1}{\epsilon} - 1$.
Now, $\forall x \in (0,1)$, $\frac{1}{\epsilon} - 1 > \frac{1}{\epsilon} - \frac{1}{x}$.
$\therefore \forall x \in (0,1)$, $\forall n \geq N$ we have $|g_n(x) - g(x)| < \epsilon$. Thus the sequence $\{g_n\}$ converges uniformly to g on $(0,1)$.

Example 14: For each $n \in \mathbb{N}$, suppose that $f_n : [0,,1] \to \mathbb{R}$ is defined by $f_n(x) = \frac{x}{nx+1}$ $\forall x \in [0,1]$.
Obviously the limit function f of the sequence of functions $\{f_n\}$ on $[0,1]$ is given by $f(x) = 0$ $\forall x \in [0,1]$.
Proceeding in a similar fashion as Ex.13 one can prove that the sequence $\{f_n\}$ converges uniformly to f on $(0,1]$. We shall establish uniform convergence of $\{f_n\}$ to f on $[0,1]$.
Let $\epsilon > 0$ be a given real.
By uniform convergence of $\{f_n\}$ to f on $(0,1]$ let us choose a positive integer N_1 such that $|f_n(x) - f(x)| < \epsilon$ $\forall x \in (0,1], \forall n \geq N_1$.
Since $f_n(0) \to f(0)$ as $n \to \infty$, let us choose a positive integer N_2 such that $|f_n(0) - f(0)| < \epsilon$ $\forall n \geq N_2$.
Set $N = max\{N_1, N_2\}$.
Then $\forall x \in [0,1]$, $\forall n \geq N$ we have $|f_n(x) - f(x)| < \epsilon$. Thus $\{f_n\}$ converges uniformly to f on $[0,1]$.
Alternative method: To prove uniform convergence of the sequence of functions $\{f_n\}$ to f on $[0,1]$ we shall use Ex.13 directly.
Let $\epsilon > 0$ be a given real.
Since by Ex.13 $\{f_n\}$ converges uniformly to f on $(0,1)$ it is possible to choose a positive integer N_0 such that $\forall x \in (0,1)$, $\forall n \geq N_0$ we have $|f_n(x) - f(x)| < \epsilon$.
Again since $f_n(0) \to f(0)$ and $f_n(1) \to f(1)$ as $n \to \infty$ one can choose two positive integers N_1, N_2 such that

$$\forall n \geq N_1, \quad |f_n(0) - f(0)| < \epsilon$$

$$\forall n \geq N_2, \quad |f_n(1) - f(1)| < \epsilon$$

.
Set $N = max\{N_0, N_1, N_2\}$. Then $\forall x \in [0,1]$, $\forall n \geq N$ we have $|f_n(x) - f(x)| < \epsilon$.
Thus $\{f_n\}$ converges uniformly to f on $[0,1]$.

Probelm1: Let S be a proper subset of $\mathbb{R}$ and F be a non-empty finite subset of $\mathbb{R} - S$. If a sequence of functions $\{f_n\}$ on $S \cup F$ converges pointwise to f on $S \cup F$ and the convergence is uniform on S then the convergence is also uniform on $S \cup F$.

Example15: For each $n \in \mathbb{N}$, let us define $h_n : \{x : x \geq 0\} \to \mathbb{R}$ by $h_n(x) = \frac{n}{x+n}$ $\forall x \geq 0$.
Then the limit function h of the sequence of functions $\{h_n\}$ on $\{x : x \geq 0\}$ is given by $h(x) = 1$ $\forall x \geq 0$.

Now to test whether $\{h_n\}$ converges uniformly to h on $\{x : x \geq 0\}$ or not :

Let $\epsilon > 0$ be a given real such that $\epsilon < 1$.

Then $\forall x \geq 0$, $|h_n(x) - h(x)| = |\frac{n}{x+n} - 1| = \frac{x}{x+n}$.

$\therefore \forall x \geq 0$ $|h_n(x) - h(x)| < \epsilon$ if $n > \frac{x(1-\epsilon)}{\epsilon}$ ....(*)

This shows that there is no positive integer N such that $|h_n(x) - h(x)| < \epsilon \; \forall x \in \{x : x \geq 0\}$, $\forall n \geq N$.

$\therefore$ the sequence $\{h_n\}$ fails to converge uniformly to h on $\{x : x \geq 0\}$.

<u>NOTE</u>: In the above example, the convergence is uniform on any bounded subset of $\{x : x \geq 0\}$. For, consider a bounded subset S of $\{x : x \geq 0\}$ and K be an upper bound of S.

For the chosen ϵ , $0 < \epsilon < 1$ and have got as (*) $|h_n(x) - h(x)| < \epsilon \; \forall x \in S$ whenever $n > \frac{x(1-\epsilon)}{\epsilon}$.

Now let us choose a positive integer N such that $N > \frac{K(1-\epsilon)}{\epsilon}$. Then $\forall n \geq N$, we have $n > \frac{x(1-\epsilon)}{\epsilon}$ $\forall x \in S$.

$\therefore |h_n(x) - h(x)| < \epsilon \; \forall x \in S, \forall n \geq N$.

Thus $\{h_n\}$ converges uniformly to h on S (Justify). In particular, the sequence $\{h_n\}$ converges uniformly to h on $[0, K]$ for any $K > 0$.

<u>Problem</u>2: Suppose that $f : S \to \mathbb{R}$ be the pointwise limit of a sequence of functions $\{f_n\}$ on S. If for each $\epsilon, 0 < \epsilon < 1$ there is a positive integer N such that $|f_n(x) - f(x)| < \epsilon \;\; \forall x \in S, \forall n \geq N$ then show that $\{f_n\}$ converges uniformly to f on S.

<u>Example</u>16: For each n, let us define $f_n : (0, 1] \to \mathbb{R}$ by $f_n(x) = \frac{1}{nx+1} \;\; \forall x \in (0, 1]$. Then the limit function f of $\{f_n\}$ on (0,1] is given by $f(x) = 0 \;\; \forall x \in (0, 1]$. To test whether $\{f_n\}$ converges uniformly to f on (0,1] or not :

Let $\epsilon > 0$ be a given real number such that $\epsilon < 1$. Now $\forall x \in (0, 1]$, $|f_n(x) - f(x)| = \frac{1}{nx+1} < \epsilon$ if $n > \frac{1}{x}(\frac{1}{\epsilon} - 1)$. Clearly there is no positive integer N such that $N > \frac{1}{x}(\frac{1}{\epsilon} - 1) \;\; \forall x \in (0, 1]$. Therefore there is no positive integer N with the property $|f_n(x) - f(x)| < \epsilon \;\; \forall x \in (0, 1]$ whenever $n \geq N$. Therefore $\{f_n\}$ does not converge uniformly to f on (0,1].

Note: The convergence is uniform in$[a, 1]$ for any a satisfying $0 < a < 1$.

<u>NOTE</u>: In the above example, the convergence is uniform on any subset T of [0,1], that has a positive lower bound. For if $a > 0$ be a lower bound of such a T then for the choosen $\epsilon(0 < \epsilon < 1)$ we would get $|f_n(x) - f(x)| < \epsilon \;\; \forall x \in T$ whenever $n > \frac{1}{x}(\frac{1}{\epsilon} - 1)$. Now let us choose a positive integer N such that $N > \frac{1}{a}(\frac{1}{\epsilon} - 1)$. Then $\forall x \in T$, $\forall n \geq N$ we have $|f_n(x) - f(x)| < \epsilon$.

Therefore $\{f_n\}$ converges uniformly to f on T. In particular, the sequence $\{f_n\}$ converges uniformly to f on $[a, 1]$ for any a satisfying $0 < a < 1$.

<u>Example</u>17: For each n, let us define $f_n : [0, 1] \to \mathbb{R}$ by $f_n(x) = \frac{nx^2}{nx+1} \;\; \forall x \in [0, 1]$. Then the limit function f of $\{f_n\}$ on [0,1] is given by $f(x) = x \;\; \forall x \in [0, 1]$. To test whether $\{f_n\}$ converges

uniformly to f on [0,1] or not :

Let $\epsilon > 0$ be a given real number such that $\epsilon < 1$. Then $\forall x \in (0,1]$, $|f_n(x) - f(x)| = \frac{x}{nx+1} < \epsilon$ if $n > \frac{1}{\epsilon} - \frac{1}{x}$.Let us choose a positive integer N such that $N > \frac{1}{\epsilon} - 1$. Then obviously $\forall x \in (0,1]$, $\forall n \geq N$ we have $|f_n(x) - f(x)| < \epsilon$.

Therefore $\{f_n\}$ converges uniformly to f on $(0,1]$ and hence on [0,1] also.

Note: See Ex.13, Problems 1 and 2 .

Example18: To test the uniform convergence of $\{f_n\}$ defined in Example 10 on $\{x : x > 0\}$.

Let $\epsilon > 0$ be a given real number such that $\epsilon < 1$. Note that $\forall x > 0$, $|f_n(x) - f(x)| = |x^2 + \frac{x^2}{1+x^2}(1 + \frac{1}{1+x^2} + ... + \frac{1}{(1+x^2)^{(n-1)}}) - (1+x^2)| = |\frac{x^2}{1+x^2}.\frac{1-\frac{1}{(1+x^2)^n}}{1-\frac{1}{1+x^2}} - 1| = \frac{1}{(1+x^2)^n} < \epsilon$ if $\frac{1}{\epsilon} < (1+x^2)^n$ i.e. if $n > \frac{log(\frac{1}{\epsilon})}{log(1+x^2)}$. Clearly there is no positive integer N such that $N > \frac{log(\frac{1}{\epsilon})}{log(1+x^2)} \quad \forall x > 0$.Therefore there is no positive integer N with the property $|f_n(x) - f(x)| < \epsilon \ \ \forall x > 0, \ \forall n \geq N$.

Thus the sequence $\{f_n\}$ does not converge uniformly to f on $\{x : x > 0\}$.

Note:(1) the convergence in the above example is uniform on any nonempty subset of $\{x : x > 0\}$, that has a positive lower bound.

(2) $\{f_n\}$ in Ex.10 cannot converge uniformly on $\{x : x \geq 0\}$. In fact, $\{f_n\}$ in Ex.10 cannot converge uniformly on any interval containing origin.

Problem:3 If a sequence of functions $\{f_n\}$ on $S \subset \mathbb{R}$ converges uniformly to $f : S \to \mathbb{R}$ then for any nonempty subset T of S the sequence $\{g_n\}$ on T defined by $g_n = f_n|T \ \ \forall n$, converges uniformly to $g = f|T$.

Example19:To test the uniform convergence of $\{f_n\}$ defined in Example 1 on $(0,1)$.

Let $\epsilon > 0$ be a given real number such that $\epsilon < 1$. Then $\forall x \in (0,1)$, $|f_n(x) - f(x)| = x^n < \epsilon$ if $\frac{1}{\epsilon} < (\frac{1}{x})^n$ or if $log\frac{1}{\epsilon} < nlog\frac{1}{x}$ or if $n > \frac{log\frac{1}{\epsilon}}{log\frac{1}{x}}$.

Note that as $x \to 1$ from left of 1 , $\frac{1}{x} \to 1$ from right of 1. Now $\frac{1}{\epsilon} > 0 (\because 0 < \epsilon < 1)$ and $log\frac{1}{x} \to \infty$ as $x \to 1 - 0$. So there is no positive integer N such that $N > \frac{log\frac{1}{\epsilon}}{log\frac{1}{x}} \quad \forall x \in (0,1)$.Therefore there is no positive integer N such that $|f_n(x) - f(x)| < \epsilon \ \ \forall x \in (0,1), \ \ \forall n \geq N$.

Thus the sequence $\{f_n\}$ does not converge uniformly to f on $(0,1)$.

Note: (1) The convergence in Ex.1 is not uniform on [0,1].(Recall problem 3).

(2) The convergence in Ex.1 is uniform on $[0,a]$ for any a satisfying $0 < a < 1$.

Example20: To test the uniform convergence of $\{f_n\}$ defined in Example 2 on $\{x : x > 0\}$.

Let $\epsilon > 0$ be a given real number such that $\epsilon < 1$. Now $\forall x > 0$, $|f_n(x) - f(x)| = \frac{1}{1+n^2x^2} < \epsilon$ if $\frac{1}{\epsilon} < 1 + n^2x^2$ or if $n > \frac{\sqrt{\frac{1}{\epsilon}-1}}{x}$.

Now as $x \to 0^+$, $\frac{\sqrt{\frac{1}{\epsilon}-1}}{x} \to \infty$ and hence there is no positive integer N such such that $\forall x > 0$ and any $n \geq N$, $\ |f_n(x) - f(x)| < \epsilon$.

Thus the sequence $\{f_n\}$ can not converge uniformly to f on $\{x : x > 0\}$.

Note: (1) The convergence in the above example is uniform on any non-empty subset of $\{x : x > 0\}$ that has a lower bound.

(2) $\{f_n\}$ in Ex.2 cannot converge uniformly on $\{x : x \geq 0\}$. In fact, $\{f_n\}$ in Ex.2 cannot converge uniformly on any interval containing origin.

THEOREM1: Suppose that $\{f_n\}$ is a sequence of functions on $S \subset \mathbb{R}$ with $f : S \to \mathbb{R}$ as its pointwise limit. Set $M_n = \sup\{|f_n(x) - f(x)| : x \in S\}$. Then the convergence of $\{f_n\}$ is uniform on S iff $\lim M_n = 0 \ \ as \ \ n \to \infty$.

PROOF: Let us suppose that $\lim M_n = 0 \ \ as \ \ n \to \infty$. To prove $\{f_n\}$ converges to f uniformly on S.

Let $\epsilon > 0$ be a given real. We choose a positive integer N such that $M_n < \epsilon \quad \forall n \geq N$ (Note that $M_n \geq 0 \ \ \forall n$).

$$\therefore \sup\{|f_n(x) - f(x)| : x \in S\} < \epsilon \quad \forall n \geq N$$

$$\therefore |f_n(x) - f(x)| < \epsilon \quad \forall x \in S, \quad \forall n \geq N$$

Thus $\{f_n\}$converges to f uniformly on S.

Conversely suppose that $\{f_n\}$converges uniformly to f on S. To prove $\lim M_n = 0 \ \ as \ \ n \to \infty$.

Let $\epsilon > 0$ be a given real. Let us choose a positive integer N such that

$$|f_n(x) - f(x)| < \epsilon/2 \quad \forall x \in S, \quad \forall n \geq N$$

$$\therefore \sup\{|f_n(x) - f(x)| : x \in S\} \leq \epsilon/2 < \epsilon \quad \forall n \geq N$$

$$\therefore M_n < \epsilon \ \ \forall n \geq N$$

Since $M_n \geq 0$, from above it follows that $\lim M_n = 0 \ \ as \ \ n \to \infty$.

COROLLARY:Suppose that $f : S \to \mathbb{R}$ is uniform limit of a sequence of functions $\{f_n\}$ on S.Then

(i) f is bounded if each f_n is bounded.

(ii)f is unbounded if each f_n is unbounded.

Proofs follow immediately from above theorem and triangle inequality.

Look at all examples which have already been proved to converge uniformly on some subset of $\mathbb{R}$ to verify corollary (i) and (ii).

Example21: To test the uniform convergence of $\{f_n\}$ defined in Example 5.

By corollary (i) above, it is clear that the convergence of $\{f_n\}$ is not uniform on $(0, 1)$.

Problem4: Cite an example of a sequence of functions $\{f_n\}$ on some $S \subset \mathbb{R}$ such that each f_n is unbounded on S but the limit function is bounded on S.

Example22: Converse of corollary (i) is false.

Consider the sequence of functions $\{f_n\}$ in example 1. Recall that its pointwise limit is bounded

and for each n, f_n is bounded.But the convergence is not uniform on $[0,1]$.
[$\because$ in example 1, we have proved that the convergence is not uniform in $(0,1)$.(Recall problem 1)].
It is also follows from Theorem 1 since, in our example, $\forall n,\ \ M_n = \sup\{|f_n(x) - f(x)| : x \in [0,1]\} = 1$ so that $\lim M_n = 1 \neq 0$.

Example23:Converse of corollary (ii) is false.

Consider the sequence of functions $\{f_n\}$ in example 10.Clearly each f_n and also the limit function f is unbounded on $\mathbb{R}$ [and also on $\{x : x > 0\}$].
In example 18, we have shown that the convergence is not uniform on $\{x : x > 0\}$] and hence also on $\mathbb{R}$.
The fact that the sequence $\{f_n\}$ cannot converge uniformly on $\{x : x > 0\}$] also follows from Theorem 1. Since,$\forall n,\ \ M_n = \sup\{|f_n(x) - f(x)| : x > 0\} = \sup\{\frac{1}{(1+x^2)^n} : x > 0\} = 1$ (Verify!). So that $\lim M_n = 1 \neq 0$.
Note: That $\{f_n\}$ in Example 10 does not converge uniformly on $\mathbb{R}$ also follows directly by using Theorem 1.

Example24: The sequence $\{f_n\}$ in Example 17 has been proved to converge uniformly on $[0,1]$. The proof also follows by Theorem 1.
Note that, $\forall n\ \ \ M_n = \sup\{|f_n(x) - f(x)| : 0 \leq x \leq 1\} = \sup\{\frac{x}{(1+nx)} : 0 \leq x \leq 1\} = \frac{1}{n+1}$.
$\therefore \lim M_n = 0$ and hence $\{f_n\}$ converges uniformly to f on $[0,1]$.

Example25: The sequence $\{f_n\}$ in Example 19 has been proved to converge pointwise but not uniformly in $\{x : x > 0\}$. The proof also follows by Theorem 1.
Note that, $\forall n\ \ \ M_n = \sup\{|f_n(x) - f(x)| : x > 0\} = \sup\{\frac{1}{(1+n^2x^2)} : x > 0\} = 1$(Verify!).
$\therefore \lim M_n = 1 \neq 0$ and hence $\{f_n\}$ does not converge uniformly to f on $\{x : x > 0\}$.

Example26:To test whether the sequence $\{f_n\}$ in example converges uniformly or not:
Recall that $\forall n,\ f_n(x) = \begin{cases} 1 - nx^2 & \text{if } x \in [0, \frac{1}{\sqrt{n}}] \\ 0 & \text{if } x \in (\frac{1}{\sqrt{n}}, 1] \end{cases}$
and the limit function f is given by $f(x) = \begin{cases} 0 & \text{if } x \in (0,1] \\ 1 & \text{if } x = 0 \end{cases}$
Obviously $\ \forall n,\ M_n = \sup\{|f_n(x) - f(x)| : 0 \leq x \leq 1\} \geq |f_n(\frac{1}{2\sqrt{n}}) - f(\frac{1}{2\sqrt{n}})| = \frac{3}{4}$
$\therefore \{M_n\}$ cannot converge to 0 and hence $\{f_n\}$ does not converge uniformly to f on $[0,1]$.

Example27: For each n, consider $f_n : \mathbb{R} \to \mathbb{R}$ defined by $f_n(x) = \frac{x^n}{n}\ \ \forall x \in \mathbb{R}$. To test whether $\{f_n\}$ converges uniformly on $\mathbb{R}$ or not.
Clearly the limit function of $\{f_n\}$ on $\mathbb{R}$ is given by $f(x) = 0\ \ \ \forall x \in \mathbb{R}$.
Note that for each $n,\ M_n = \sup\{|f_n(x) - f(x)| : x \in \mathbb{R}\} = \sup\{\frac{x^n}{n} : x \in \mathbb{R}\} \geq \frac{(\sqrt{n})^2}{n} = 1$.
$\therefore \{M_n\}$ cannot converge to 0 and hence the sequence $\{f_n\}$ cannot converge uniformly to f on $\mathbb{R}$.

Example28: In example 11, the sequence $\{f_n\}$ has been proved to converge uniformly on $\mathbb{R}$.The proof also follows from Theorem 1.
Note that $\forall n, \quad f_n(x) = \frac{sinnx}{n} \quad \forall x \in \mathbb{R}$ and the limit function f is given by $f(x) = 0 \quad \forall x \in \mathbb{R}$.
$\therefore\ \forall n,\ M_n = \sup\{|f_n(x) - f(x)| : x \in \mathbb{R}\} = \sup\{|\frac{sinnx}{n}| : x \in \mathbb{R}\} \leq \frac{1}{n}$.
As $M_n \geq 0 \ \ \forall n$, from above it follows that the sequence $\{M_n\}$ converges to 0. Hence that convergence of the sequence of functions $\{f_n\}$ is uniform on $\mathbb{R}$.
Example29: Consider the sequence of functions $\{f_n\}$ in example 12. Proceeding as Example 28 one can easily check that $M_n = \sup\{|f_n(x) - f(x)| : x \in \mathbb{R}\} \leq \frac{1}{\sqrt{n}}$ so that M_n converges to 0 (as $M_n \geq 0\ \forall n$).Thus the convergence of $\{f_n\}$ is uniform on $\mathbb{R}$.
Example30: To test uniform convergence of the sequence of functions $\{\frac{nx}{1+n^2x^2}\}$ on the interval $[0,1]$. $\forall n$, let us define $f_n : [0,1] \to \mathbb{R}$ by $f_n(x) = \frac{nx}{1+n^2x^2} \ \ \forall x \in [0,1]$. Obviously for any $x \in [0,1], \ \ f_n(x) = \frac{\frac{x}{n}}{\frac{1}{n^2}+x^2} \to 0 \quad$ as $\ n \to \infty$ so that the limit function $f : [0,1] \to \mathbb{R}$ of the sequence of functions $\{f_n\}$ is given by $f(x) = 0 \ \ \forall x \in [0,1]$. Note that $\forall n \in \mathbb{N}, \ \ \forall x \in [0,1]$ applying G.M. $\leq$ A.M. we have $nx \leq \frac{1+n^2x^2}{2}$ or $\frac{nx}{1+n^2x^2} \leq \frac{1}{2}$ and the equality occurs when $1 = nx$ that is when $x = \frac{1}{n}$.
Now $\ \forall n \ \ M_n = \sup\{|f_n(x) - f(x)| : 0 \leq x \leq 1\} = \sup\{\frac{nx}{1+n^2x^2} : 0 \leq x \leq 1\}(= \frac{n.\frac{1}{n}}{1+n^2\frac{1}{n^2}}) = \frac{1}{2}$
$[\therefore \max\{\frac{nx}{1+n^2x^2} : 0 \leq x \leq 1\} = \frac{1}{2}$ that occurs at $\frac{1}{n} \in [0,1]]$.
$\therefore \lim M_n = \frac{1}{2} \neq 0$. Thus the given sequence of functions fails to converge uniformly on $[0,1]$.
Example31: To test uniform convergence of the sequence of functions $\{f_n\}$ on $[0,1]$where $\forall n, \ \ f_n :$ $[0,1] \to \mathbb{R}$ is defined by $f_n(x) = \begin{cases} nx^2 & \text{if } x \in [0,\frac{1}{n}) \\ x & \text{if } x \in [\frac{1}{n},1] \end{cases}$
Let $f : [0,1] \to \mathbb{R}$ be the limit function. Clearly $f(0) = \lim f_n(0) = 0; f(1) = \lim f_n(1) = 1$. Let $x_0 \in (0,1)$. Choose $N \in \mathbb{N}$ such that $\frac{1}{N}x_0$. then $x_0 \in [\frac{1}{n},1] \ \forall n \geq N$ so that $f_n(x_0) = x_0 \ \ \forall n \geq N$.
$\therefore\ f(x_0) = \lim f_n(x_0) = x_0$. Since $x_0 \in (0,1)$ is arbitrary it follows that $f(x) = x \ \ \forall x \in (0,1).\therefore$ $f(x_0) = \lim f_n(x_0) = x_0$. Combining the above results we get $f(x) = x \ \ \forall x \in [0,1]$.
Now to test the uniform convergence of $\{f_n\}$ on $[0,1]$ let us define $\forall n, \ \ M_n = \sup\{|f_n(x) - f(x)| :$ $0 \leq x \leq 1\}$.Clearly $\forall n,\ |f_n(x) - f(x)| = \begin{cases} x - nx^2 & \text{if } x \in [0,\frac{1}{n}) \\ 0 & \text{if } x \in [\frac{1}{n},1] \end{cases}$
Let us define $\phi_n : (0,\frac{1}{n}) \to \mathbb{R}$ by $\phi_n(x) = x - nx^2 \ \ \forall x \in (0,1)$. Obviously ϕ_n is differentiable on $(0,1)$ and $\forall x \in (0,1), \ \ \phi'_n(x) = 1 - 2nx = 2n(\frac{1}{2n} - x)$. $\therefore\ \phi'_n(x)$ changes sign from positive to negative as x crosses the point $x = \frac{1}{2n}$ from left to right and ϕ_n is continuous at $x = \frac{1}{2n}$. So $\phi_n(x)$ has global maximum at $x = \frac{1}{2n}$.
So we have, $|f_n(x) - f(x)| = \begin{cases} 0 & \text{if } x = 0 \\ \phi_n(x) & \text{if } x \in (0,\frac{1}{n}) \\ 0 & \text{if } x \in [\frac{1}{n},1] \end{cases}$ and hence $\forall n, \ \ M_n = \sup\{|f_n(x) - f(x)| :$ $0 < x < 1\} = \max\{\phi(x) : 0 < x < \frac{1}{n}\} = \phi_n(\frac{1}{2n}) = \frac{1}{2n} - n\frac{1}{4n^2} = \frac{1}{4n}$

Now $\{M_n\}$ converges to 0 ($\because M_n = \frac{1}{4n}\ \forall n$) and the given sequence of functions converges uniformly on $[0,1]$.

Example32: To test uniform convergence of the sequence of functions $\{x^n(1-x)\}$ on the interval $[0,1]$. For each n, let us define $f_n : [0,1] \to \mathbb{R}$ by $f_n(x) = x^n(1-x)\ \ \forall x \in [0,1]$. Let $f : [0,1] \to \mathbb{R}$ be the limit function. Clearly $\forall x \in [0,1],\ f(x) = \lim f_n(x) = 0$.

Now to test uniform convergence of $\{f_n\}$ on $[0,1]$ let us define $\forall n,\ \ M_n = \sup\{|f_n(x) - f(x)| : 0 \leq x \leq 1\} = \sup\{x^n - x^{n+1} : 0 \leq x \leq 1\}$. Consider $\phi_n : [0,1] \to \mathbb{R}$ defined by $\phi_n(x) = x^n - x^{n+1}\ \ \forall x \in [0,1]$.Obviously ϕ_n is differentiable on $[0,1]$ and $\forall x \in [0,1]\quad \phi_n'(x) = x^{n-1}[n-(n+1)x]$.

$\therefore \phi_n'$ changes sign from positive to negative as x crosses the point $x = \frac{n}{n+1}$ from left to right and ϕ_n is continuous. So ϕ_n has global maximum at $x = \frac{n}{n+1}$.

$\therefore\ \forall n,\ M_n = \sup\{\phi_n(x) : 0 \leq x \leq 1\} = \phi_n(\frac{n}{n+1}) = \frac{1}{(1+\frac{1}{n})^n}\frac{1}{n+1}$.

Since $\lim(1+\frac{1}{n})^n = e \neq 0$ and $\lim\frac{1}{n+1} = 0$ it follows that $M_n \to 0$ as $n \to \infty$. Hence the given sequence of functions converges uniformly on $[0,1]$.

Example33: Suppose that $\psi : [0,1] \to \mathbb{R}$ is a continuous function with $\psi(1) = 0$ and $\{f_n\}$ is a sequence of functions on $[0,1]$ defined by, $f_n(x) = x^n\psi(x)\ \ \forall x \in [0,1], \forall n$. To test the uniform convergence of $\{f_n\}$ on $[0,1]$. Let f be the limit function on $[0,1]$. Clearly $f(0) = \lim f_n(0) = 0$; $f(1) = \lim f_n(1) = 0,\ \because \psi(1) = 0$. Now let $x_0 \in (0,1)$. Since ψ is continuous on $[0,1]$, it is bounded on $[0,1]$. Let us choose $K > 0$ such that $|\psi(x)| \leq K\ \forall x \in [0,1]$. Now $|f_n(x_0)| = |x_0^n.\psi(x_0)| = x_0^n|\psi(x_0)| \leq x_0^n.K$. Since $x_0 \in (0,1)$ was arbitrary it follows that $f(x) = 0\ \forall x \in (0,1)$.

Thus we get the limit function f of the sequence of functions $\{f_n\}$ on $[0,1]$ as $f(x) = 0\ \forall x \in [0,1]$.

To test uniform convergence of the sequence $\{f_n\}$ on $[0,1]$.

Case (i): $\psi(x) = 0\ \forall x \in [0,1]$.

Then $\forall n,\ f_n(x) = 0\ \forall x \in [0,1]$ and hence the sequence $\{f_n\}$ converges uniformly to f trivially.

Case (ii): $\psi(x) \neq 0$ for some $x \in [0,1]$.

Set $\forall n,\ M_n = \sup\{|f_n(x) - f(x)| : 0 \leq x \leq 1\} = \sup\{x^n|\psi(x)| : 0 \leq x \leq 1\}$. Now $\forall n$, by continuity of the function $x^n.|\psi(x)|$ on $[0,1]$ let us choose $x_n \in [0,1]$ such that $M_n = x_n^n|\psi(x_n)|$. Clearly $M_n > 0\ \forall n$ as $\psi(x) \neq 0$ for some $x \in [0,1]$ and hence $x_n \neq 0, x_n \neq 1\ \forall n$. Let $\sup x_n = a$. Then $0 < a \leq 1$. Let us consider the case $a < 1$. Since $x_n \leq a\ \forall n$ we have $M_n = x_n^n|\psi(x_n)| \leq a^n.K$ and hence $\lim M_n = 0 (\because \lim a^n = 0\ as\ 0 < a < 1)$.

Next let us consider the case $a = 1$. Let us choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to 1$ as $k \to \infty$.

Note that $\forall k,\ M_{n_k} = x_{n_k}^{n_k}|\psi(x_{n_k})| < |\psi(x_{n_k})|$ and by continuity of $|\psi(x)|$ at $x = 1$ we have $\lim_{k\to\infty}|\psi(x_{n_k})| = |\psi(1)| = 0\quad$ since $\ \psi(1) = 0$.

$\therefore \lim_{k\to\infty} M_{n_k} = 0$.

Thus 0 is a cluster point of the sequence$\{M_n\}$. But as $\forall x \in [0,1], x \geq x^2 \geq x^3 \geq x^4 \geq \cdots$ we have $x|\psi(x)| \geq x^2|\psi(x)| \geq x^3|\psi(x)| \geq x^4|\psi(x)| \geq \cdots$. Therefore $M_1 \geq M_2 \geq M_3 \geq M_4 \geq \cdots$.
$\therefore \{M_n\}$ is a monotone decreasing sequence having a cluster point converges to that cluster point , it follows that $\lim_{n\to\infty} M_n = 0$. Hence $\{f_n\}$ converges uniformly on $[0,1]$.
<u>THEOREM</u>2 (Cauchy criterion for uniform convergence of a sequence of functions): A sequence of functions $\{f_n\}$, defined on $S \subset \mathbb{R}$, converges uniformly on S if and only if for every $\epsilon > 0$ there exists a positive integer N such that $|f_m(x) - f_n(x)| < \epsilon \quad \forall x \in S,\ \forall m, n \geq N$.
PROOF : Suppose that $\{f_n\}$ converges uniformly on S. Let $f : S \to \mathbb{R}$ be the uniform limit.
Let $\epsilon > 0$ be a given real. Let us choose a positive integer N such that

$$|f_k(x) - f(x)| < \epsilon/2 \quad \forall x \in S,\ \forall k \geq N$$

. Therefore $\forall x \in S,\ \forall m, n \geq N$ we have
$|f_m(x) - f_n(x)| \leq |f_m(x) - f(x)| + |f_n(x) - f(x)| < \epsilon/2 + \epsilon/2 = \epsilon$.
Conversely, suppose that the given Cauchy condition is satisfied for the sequence of functions $\{f_n\}$ on S. To prove : $\{f_n\}$ converges uniformly on S.
Fix $x \in S$. Clearly from the given condition, $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is Cauchy-complete $\lim_{n\to\infty} f_n(x)$ belongs to $\mathbb{R}$. Since $x \in S$ is arbitrary it follows that $\lim_{n\to\infty} f_n(x)$ exists in $\mathbb{R}$ $\forall x \in S$. Let us define $f : S \to \mathbb{R}$ by $f(x) = \lim_{n\to\infty} f_n(x)$ $\forall x \in S$.We shall now prove that this f is the uniform limit of $\{f_n\}$ on S.
Let $\epsilon > 0$ be a given real. From the given condition, let us choose a positive integer N such that $|f_m(x) - f_n(x)| < \epsilon/2 \quad \forall x \in S,\ \forall m, n \geq N$.
keeping $n \geq N$ fixed, for each $x \in S$ we have $\lim_{m\to\infty} |f_m(x) - f_n(x)| \leq \epsilon/2$ or, $|f(x) - f_n(x)| \leq \epsilon/2$.
Therefore $|f_n(x) - f(x)| < \epsilon$ $\forall x \in S$.
Since $n \geq N$ was arbitrary it follows that $|f_n(x) - f(x)| < \epsilon$ $\forall x \in S,\ \forall n \geq N$.
Thus $\{f_n\}$ converges uniformly to f on S.
NOTE: Theorem 2 can be restated as the following :
A sequence of functions $\{f_n\}$, defined on $S \subset \mathbb{R}$, converges uniformly on S if and only if for every $\epsilon > 0$ there exists a positive integer N such that

$$\text{Sup}\{|f_m(x) - f_n(x)| : x \in S\} < \epsilon \quad \forall m, n \geq N$$

.

<u>Example</u>34: the sequence of functions $\{f_n\}$ in example 11has already been proved to converge uniformly on $\mathbb{R}$ (See Example 28 also). The proof also follows from theorem 2.
Note that $\text{Sup}\{|f_m(x) - f_n(x)| : x \in \mathbb{R}\} = \text{Sup}\{|\frac{Sinmx}{m} - \frac{Sinnx}{n}| : x \in \mathbb{R}\} \leq \text{Sup}\{|\frac{Sinmx}{m}| : x \in \mathbb{R}\} + \text{Sup}\{|\frac{Sinnx}{n}| : x \in \mathbb{R}\} \leq \frac{1}{m} + \frac{1}{n}$.

Let $\epsilon > 0$ be a given real.
Let us choose a positive integer N such that $\frac{1}{N} < \epsilon/2$. Then $\forall m, n \geq N$ we have $\frac{1}{m} < \epsilon/2,\ \frac{1}{n} < \epsilon/2$ and hence $\text{Sup}\{|f_m(x) - f_n(x)| : x \in \mathbb{R}\} < \epsilon \quad \forall m, n \geq N$.
Thus $\{f_n\}$ converges uniformly on $\mathbb{R}$.

Example35: The sequence of functions $\{f_n\}$ in example 12 (see example 29 also) can be proved to converge uniformly on $\mathbb{R}$ in a similar way as Example 34.

Example36: The sequence of functions $\{f_n\}$ in Example 1 converges uniformly on $[0, a]$ for any a satisfying $0 < a < 1$ but does not converge uniformly on $[0, 1]$(See Example 19,22). This can also be proved using Theorem 2.
Let $0 < a < 1$. Note that $\text{Sup}\{|x^m - x^n| : x \in [0, a]\} \leq \text{Sup}\{x^m : x \in [0, a]\} + \text{Sup}\{x^n : x \in [0, a]\} \leq a^m + a^n$.
Let $\epsilon > 0$ be a given real. We choose a positive integer N such that $a^N < \epsilon/2$. Then $\forall m, n \geq N,\ a^m < \epsilon/2,\ a^n < \epsilon/2$ so that $\text{Sup}\{|x^m - x^n| : x \in [0, a]\} < \epsilon \ \forall m, n \geq N$.
Therefore the sequence $\{f_n\}$ converges uniformly on $[0, a]$.
Since a in $(0, 1)$ was arbitrary it follows that the sequence of functions $\{f_n\}$ converge uniformly on $[0, a]$ for any a satisfying $0 < a < 1$.
In order to prove that the sequence of functions $\{f_n\}$ does not converge uniformly on $[0, 1]$ it is sufficient to establish that the convergence is not uniform in $(0, 1)$. For this we shall show that $\lim_{n\to\infty} \text{Sup}\{|f_{2n}(x) - f_n(x)| : 0 < x < 1\} \neq 0$.
Set $\phi_n(x) = |f_{2n}(x) - f_n(x)| \quad \forall x \in (0, 1)$.
Therefore $\phi_n(x) = x^n - x^{2n} \quad \forall x \in (0, 1)$.
Now $\phi_n'(x) = nx^{n-1}(1 - 2x^n)$. This shows that $\phi_n'(x)$ changes sign from positive to negative as x crosses the point $(\frac{1}{2})^{\frac{1}{n}}$ from left to right and ϕ_n is continuous at $x = (\frac{1}{2})^{\frac{1}{n}}$. So ϕ_n has global maximum at $x = (\frac{1}{2})^{\frac{1}{n}}$.
Therefore $\text{Sup}\{|f_{2n}(x) - f_n(x)| : 0 < x < 1\} = \phi_n((\frac{1}{2})^{\frac{1}{n}}) = \frac{1}{2} - (\frac{1}{2})^2 = \frac{1}{4}$.
$\lim_{n\to\infty} \text{Sup}\{|f_{2n}(x) - f_n(x)| : 0 < x < 1\} = \frac{1}{4} \neq 0$.
This shows that the condition in Cauchy criterion for uniform convergence is not satisfied by the sequence of functions $\{f_n\}$ in $(0, 1)$.
Therefore $\{f_n\}$ fails to converge uniformly in $(0, 1)$.

Example37: The sequence of functions $\{f_n\}$ in Example 16 has been proved to converge pointwise but not uniformly in $[0, 1]$. The fact that the convergence is not uniform on $[0, 1]$ can also be established using Theorem 2.
Here, as Example 36, it is sufficient to establish $\lim_{n\to\infty} \text{Sup}\{|f_{2n}(x) - f_n(x)| : 0 \leq x \leq 1\} \neq 0$.
Set $\Phi_n(x) = |f_{2n}(x) - f_n(x)| \quad \forall x \in [0, 1]$.

Then $\Phi_n(x) = \frac{1}{nx+1} - \frac{1}{2nx+1} \quad \forall x \in [0,1]$.
Note that, $\Phi'_n(x) = \frac{2n}{(2nx+1)^2} - \frac{n}{(nx+1)^2} = n\frac{2(nx+1)^2-(2nx+1)^2}{(2nx+1)^2(nx+1)^2} = n\frac{(-2n^2x^2+1}{(2nx+1)^2(nx+1)^2}$.
Therefore $\Phi'_n(x)$ changes sign from positive to negative as x crosses the point $x = \frac{1}{n\sqrt{2}}$ from left to right and Φ_n is continuous. Therefore Φ_n has aglobal maximum at the point $x = \frac{1}{n\sqrt{2}}$.
$\text{Sup}\{|f_{2n}(x) - f_n(x)| : 0 \le x \le 1\} = \Phi_n(\frac{1}{n\sqrt{2}}) = \frac{1}{\frac{1}{\sqrt{2}}+1} - \frac{1}{\sqrt{2}+1} = \frac{\sqrt{2}-1}{\sqrt{2}+1}$.
Clearly, $\lim_{n\to\infty} \text{Sup}\{|f_{2n}(x) - f_n(x)| : 0 \le x \le 1\} = \frac{\sqrt{2}-1}{\sqrt{2}+1} \ne 0$.
NOTE: That the convergence in $[a,1]$, $0 < a < 1$, is uniform can also be established by Theorem 2.
Example38: In example 21 we have proved that the sequence of functions $\{f_n\}$ in example 5 does not converge uniformly on $(0,1)$. This fact can also be proved by using Theorem 2.
Recall that $f_n(x) = 1 + x + x^2 + \cdots + x^{n-1} \quad \forall x \in (0,1)$. Now $\forall m, n$ with $m > n$ and $\forall x \in (0,1)$
$|f_m(x) - f_n(x)| = x^n + x^{n+1} + \cdots + x^{m-1} \ge x^n$.
Therefore, $\sup\{|f_m(x) - f_n(x)| : x \in (0,1)\} \ge \sup\{x^n : x \in (0,1)\} = 1$.
This suggest that the Cauchy condition cannot be satisfied by the sequence of functions $\{f_n\}$ on $(0,1)$.
Therefore $\{f_n\}$ cannot converge uniformly on $(0,1)$.
Example39: In example 18 we have proved that the sequence of functions $\{f_n\}$ in example 10 does not converge uniformly on $\{x : x > 0\}$. This fact can also be proved by using Theorem 2.
Recall that $f_n(x) = x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \cdots + \frac{x^2}{(1+x^2)^n} \quad \forall x > 0$.
Now $\forall m, n$ with $m > n$ and $\forall x > 0$
$|f_m(x) - f_n(x)| = \frac{x^2}{(1+x^2)^{n+1}} + \frac{x^2}{(1+x^2)^{n+2}} + \cdots + \frac{x^2}{(1+x^2)^m}$.
Therefore for each n and for each $x > 0$ we have
$|f_{2n}(x) - f_n(x)| > \frac{x^2}{(1+x^2)^{n+1}} + \frac{x^2}{(1+x^2)^{n+1}} + \cdots + \frac{x^2}{(1+x^2)^{n+1}}$ (n times) $= \frac{nx^2}{(1+x^2)^{n+1}} = \Phi_n(x)$ (say).
Now $\Phi'_n(x) = \frac{2nx}{(1+x^2)^{n+1}} + nx^2[-(n+1)]\frac{2x}{(1+x^2)^{n+2}}$
$= \frac{2nx}{(1+x^2)^{n+1}}[1 - \frac{(n+1)x^2}{1+x^2}]$
$= \frac{2nx}{(1+x^2)^{n+1}} \frac{(1-nx^2)}{1+x^2}$ which changes sign from positive to negative as x crosses the point $\frac{1}{\sqrt{n}}$ from left to right and Φ_n is continuous at $\frac{1}{\sqrt{n}}$. So, Φ_n has global maximum at $x = \frac{1}{\sqrt{n}}$.
Therefore $\text{Sup}\{|f_{2n}(x) - f_n(x)| : x > 0\} \ge \text{Sup}\{\Phi_n(x) : x > 0\} = \Phi_n(\frac{1}{\sqrt{n}}) = \frac{1}{(1+\frac{1}{n})^{n+1}}$.
Therefore, $\lim_{n\to\infty} \text{Sup}\{|f_{2n}(x) - f_n(x)| : x > 0\} \ge \frac{1}{e} > 0$.
Thus $\{f_n\}$ does not satisfy Cauchy condition for uniform convergence and hence the sequence of functions $\{f_n\}$ fails to converge uniformly on $\{x : x > 0\}$.
Example40: To test whether the sequence of functions $\{f_n\}$ in example 6 converges uniformly on $[0,1]$ or not:

Recall that for some ennumeration $r_1, r_2, \cdots$ of rationals in $[0,1]$, $\forall n$, f_n is given by

$$f_n(x) = \begin{cases} 1 & \text{if } x \in \{r_1, r_2, \cdots, r_n\} \\ 0 & \text{if } x \in [0,1] - \{r_1, r_2, \cdots, r_n\} \end{cases}$$

Obviously, $f_n(r_{n+1}) = 0$ but $f_{n+1}(r_{n+1}) = 1$.
Therefor, $\text{Sup}\{|f_{n+1}(x) - f_n(x)| : x \in [0,1]\} \geq 1$.
In particular, the sequence of functions $\{f_n\}$ cannot satisfy Cauchy condition for uniform convergence. Hence the sequence $\{f_n\}$ fails to converge uniformly on $[0,1]$.
<u>Problem</u>5: Using Theorem 1 prove that the sequence of functions in Example 40 does not converge uniformly on $[0,1]$.
<u>THEOREM</u>3: Suppose that $\{f_n\}$ is a sequence of functions on $S \subset \mathbb{R}$ and c is a point in S such that each f_n is continuous at c. If the sequence $\{f_n\}$ converges uniformly to $f : S \to \mathbb{R}$ on S then f is also continuous at c.
(Uniform convergence preserves continuity)
PROOF: Let $\epsilon > 0$ be a given real.
By uniform convergence of $\{f_n\}$ to f on S, let us choose a positive integer N such that

$$|f_n(x) - f(x)| < \frac{\epsilon}{3} \quad \forall x \in S, \quad \forall n \geq N \qquad .....(i).$$

Now, by continuity of f_N at c let us choose a $\delta > 0$ such that

$$|f_N(x) - f_N(c)| < \frac{\epsilon}{3} \quad \forall x \in (c-\delta, c+\delta) \cap S \qquad .....(ii).$$

Note that $\forall x \in S$

$$\begin{aligned} |f(x) - f(c)| &= |f(x) - f_N(x) + f_N(x) - f_N(c) + f_N(c) - f(c)| \\ &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(c)| + |f_N(c) - f(c)| \\ &\leq \frac{\epsilon}{3} + |f_N(x) - f_N(c)| + \frac{\epsilon}{3} \qquad \text{by}(i) \end{aligned}$$

Therefore from (ii) and the above inequality we get

$$\forall x \in (c-\delta, c+\delta) \cap S, \quad |f(x) - f(c)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

Thus f is continuous at c.
NOTE: Converse of the above theorem is not true.
<u>Example</u>41: Let us consider the sequence of functions $\{f_n\}$ on $[0,1]$ where $\forall n, \quad f_n(x) = \frac{nx}{1+n^2x^2} \quad \forall x \in [0,1]$. Note that each f_n is continuous at each point of $[0,1]$ and also the limit function which being

identically zero on $[0,1]$ is obviously continuous at each point of $[0,1]$. But in example 30 we have already seen that this sequence of functions $\{f_n\}$ is not uniformly convergent on $[0,1]$.

Corollary: Suppose that $\{f_n\}$ is a sequence of continuous functions on $S \subset \mathbb{R}$. If the sequence $\{f_n\}$ converges uniformly to $f : S \to \mathbb{R}$ then f is also continuous.

NOTE: Example 41 suggests that converse of the above corollary is not true. REMARK: Theorem 3 is often helpful to disprove uniform convergence of some sequences of functions.

From this theorem we can immediately conclude that the sequence of functions $\{f_n\}$ in each Examples 1,2,3,4,6,10 does not converge uniformly on the corresponding domain.(See note 1 just after example 10).

THEOREM4: Suppose that $\{f_n\}$ is a sequence of continuous functions on $S \subset \mathbb{R}$. If the sequence $\{f_n\}$ converges uniformly to $f : S \to \mathbb{R}$ on S then for any x in S and any sequence $\{x_n\}$ in S converging to x the sequence $\{f_n(x_n)\}$ converges to $f(x)$.

PROOF: Let us suppose that $\{f_n\}$ converges uniformly to f on S. Let $x \in S$ and $\{x_n\}$ be a sequence in S such that $\lim x_n = x$.

To prove: $\lim f_n(x_n) = f(x)$.

Let $\epsilon > 0$ be a given real. By uniform convergence of $\{f_n\}$ to f on S let us choose a positive integer N_1 such that $|f_n(x) - f(x)| < \frac{\epsilon}{2} \quad \forall x \in S \;\; \forall n \geq N_1$.

By Theorem 3, f is continuous at x. So by Continuity of f at x let us choose a $\delta > 0$ such that $|f(x) - f(y)| < \frac{\epsilon}{2} \;\; \forall y \in (x-\delta, x+\delta) \cap S$.

Since $\lim x_n = x$, we can choose a positive integer N_2 such that $x_n \in (x-\delta, x+\delta) \quad \forall n \geq N_2$.

Then $\forall n \geq max\{N_1, N_2\}$ we have

$|f_n(x_n) - f(x)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

Thus $\lim f_n(x_n) = f(x)$.

NOTE: Converse of above theorem is not true.

Example42: Consider the sequence of functions $\{f_n\}$ in Example 41. Note that each f_n is continuous on $[0,1]$ with the limit function f such that $f(x) = 0 \quad \forall x \in [0,1]$. Clearly $\forall x \in [0,1]$ and each sequence $\{x_n\}$ in $[0,1]$ converging to x we have

$f_n(x_n) = \frac{nx_n}{1+n^2x_n^2} = \frac{\frac{x_n}{n}}{\frac{1}{n^2}+x_n^2} \to \frac{0.x}{0+x} = 0 = f(x)$.

But the convergence of the sequence $\{f_n\}$ is not uniform on $[0,1]$.

REMARK: Theorem 4 is very useful to disprove uniform convergence of some sequences of functions:

Note that $0 \in [0,1]$ and $\{\frac{1}{n}$ is a sequence in $[0,1]$ converging to 0. Now $f_n(\frac{1}{n}) = \frac{1}{2} \quad \forall n$.

Therefore $\lim f_n(\frac{1}{n}) = \frac{1}{2} \neq 0 = f(0)$. So by Theorem 4 the convergence of the sequence $\{f_n\}$ to f is not uniform on $[0,1]$.

Example43: The sequence of functions $\{f_n\}$ in Example 9 does not converge uniformly on $[0,1]$.
Recall that $f_n(x) = \frac{Sinnx}{1+nx} \quad \forall x \in [0,1] \;\; \forall n$. clearly each f_n is a continuous function on $[0,1]$. since $f_n(0) = 0 \quad \forall n$ and $|f_n(x)| \leq \frac{1}{1+nx} \quad \forall x \in (0,1], \;\; \forall n$ it follows that the limit function $f : [0,1] \to \mathbb{R}$ is given by $f(x) = 0 \quad \forall x \in [0,1]$.
Note that $0 \in [0,1]$ and $\{\frac{1}{n}$ is a sequence in $[0,1]$ converging to 0. Now $f_n(\frac{1}{n}) = \frac{Sin1}{2} \quad \forall n$.
Therefore $\lim f_n(\frac{1}{n}) = \frac{Sin1}{2} \neq 0 = f(0)$.
So by Theorem 4, the sequence $\{f_n\}$ does not converge uniformly to f on $[0,1]$.
Example44: Let us consider the sequence of functions $\{f_n\}$ on $[0,1]$ given by $f_n(x) = \frac{x^2}{x^2+(1-nx)^2} \;\; \forall x \in [01], \;\; \forall n$. Clearly each f_n is continuous function on $[0,1]$. Since $f_n(0) = 0 \;\forall n$ and $f_n(x) = \frac{1}{1+(\frac{1}{x}-n)^2} \quad \forall x \in (0,1], \; \forall n$, it follows that the limit $f : [0,1] \to \mathbb{R}$ is given by $f(x) = 0 \quad \forall x \in [0,1]$.
Note that $0 \in [0,1]$ and $\{\frac{1}{n}$ is a sequence in $[0,1]$ converging to 0. Now $f_n(\frac{1}{n}) = 1 \;\forall n$.
Therefore $\lim f_n(\frac{1}{n}) = 1 \neq 0 = f(0)$.
So by Theorem 4, the sequence $\{f_n\}$ does not converge uniformly to f on $[0,1]$.
Example45: Let us consider the sequence of functions $\{f_n\}$ on $[-1,1]$ given by $f_n(x) = nxe^{-nx^2} \;\; \forall x \in [-1,1], \;\; \forall n$. Clearly each f_n is continuous function on $[-1,1]$. Let $f : [-1,1] \to \mathbb{R}$ be the limit function of $\{f_n\}$. Note that $f_n(0) = 0 \;\; \forall n$ and so $f(0) = \lim f_n(0) = 0$. Now for $x \in [-1,1] - \{0\}$, $f_n(x) = \frac{1}{x}\frac{nx^2}{e^{nx^2}} \quad \forall n$. Since for $k > 0 \quad \lim \frac{y_k}{e^{y_k}} = 0$ it follows that $\forall x \in [-1,1] - \{0\}, \;\; \lim \frac{nx^2}{e^{nx^2}} = 0$.
Therefore $\forall x \in [-1,1] - \{0\}, \quad f(x) = \lim f_n(x) = \frac{1}{x}.0 = 0$.
Thus the limit function f of the given sequence of functions is given by $f(x) = 0 \quad \forall x \in [-1,1]$.
We shall use theorem 4 to prove that the convergence of the sequence of functions $\{f_n\}$ is not uniform on $[-1,1]$.
Note that $0 \in [-1,1]$ and $\{\frac{1}{n}\}$ is a sequence in $[-1,1]$ converging to 0. Now $f_n(\frac{1}{n}) = \frac{1}{e^{\frac{1}{n}}} \quad \forall n$.
Therefore $\lim f_n(\frac{1}{n}) = 1 \neq 0 = f(0)$.
So by Theorem 4, we conclude that the sequence of functions $\{f_n\}$ does not converge uniformly on $[-1,1]$.
Problem6: Using Theorem 1 prove that the sequence of functions in Example 45 does not converge uniformly on $[-1,1]$.
Example46: Let us consider the sequence of functions $\{f_n\}$ on $[0,1]$ defined by

$$f_n(x) = \begin{cases} (1-n^2x^2)Sin\; nx & \text{for } 0 \leq x \leq \frac{1}{n} \\ 0 & \text{for } \frac{1}{n} < x \leq 1 \end{cases} \qquad \forall n$$

Clearly each $\{f_n\}$ is a continuous function on $[0,1]$. Let $f : [0,1] \to \mathbb{R}$ be the limit function of $\{f_n\}$. Note that $f_n(0) = 0 \;\; \forall n$ and so $f(0) = \lim f_n(0) = 0$. Next let $x_0 \in (0,1]$ Choose N such that $\frac{1}{N} < x_0$. Then $\frac{1}{n} < x_0 \leq 1 \;\; \forall n \geq N$.
Therefore $f_n(x_0) = 0 \;\; \forall n \geq N$. So $f(x_0) = \lim f_n(x_0) = 0$. Since x_0 is arbitrary it follows that

$f(x) = 0 \quad \forall x \in (0,1]$. Thus $f : [0,1] \to \mathbb{R}$ is given by $f(x) = 0 \ \ \forall x \in [0,1]$.
Note that $0 \in [0,1]$ and $\{\frac{1}{2n}\}$ is a sequence in $[0,1]$ converging to 0. Now $f_n(\frac{1}{2n}) = \frac{3}{4}Sin\ \frac{1}{2} \ \ \forall n.$
Therefore $\lim f_n(\frac{1}{2n}) = \frac{3}{4}Sin\ \frac{1}{2} \neq 0 = f(0)$. Thus by Theorem 4 we conclude that the sequence of functions $\{f_n\}$ does not converge uniformly on $[0,1]$.

<u>THEOREM</u>5: Suppose that $\{f_n\}$ is a sequence of Riemann integrable functions on $[a,b]$. If the sequence $\{f_n\}$ converges uniformly to $f : [a,b] \to \mathbb{R}$ on $[a,b]$ then f is also integrable on $[a,b]$ and moreover $\int_a^b f(x)dx = \lim \int_a^b f_n(x)dx$.
(Uniform convergence preserves Riemann integrability and for a uniformly convergent sequence of functions, integral of the limit function is equal to the limit of the integrals).

PROOF: Let us suppose that $f : [a,b] \to \mathbb{R}$ is the uniform limit of the sequence of Riemann integrable functions on $[a,b]$. So f is bounded on $[a,b]$, by corollary (i) of Theorem 1, since each f_n is neccessarily bounded on $[a,b]$.
Now to prove Riemann integrability of f on $[a,b]$.
Let $\epsilon > 0$ be a given real.
By uniform convergence of $\{f_n\}$ to f on $[a,b]$ let us choose a positive integer N such that
$|f_n(x) - f(x)| < \frac{\epsilon}{5(b-a)} \quad \forall x \in [a,b], \ \ \forall n \geq N$ (i)
Now by the Riemann integrability of the function f_N on $[a,b]$ we choose a partition $P := (x_0, x_1, \cdots, x_n)$ of $[a,b]$ such that $U(P, f_N) - L(P, f_N) < \frac{\epsilon}{5}$ (ii)
where $U(P, f_N) = \sum_{r=1}^n M_r(f_N)(x_r - x_{r-1}), \quad M_r(f_N) = \sup\{f_N(x) : x \in [x_{r-1} - x_r]\} \ \ \forall r = 1, 2, ..., n$
and $L(P, f_N) = \sum_{r=1}^n m_r(f_N)(x_r - x_{r-1}), \quad m_r(f_N) = \inf\{f_N(x) : x \in [x_{r-1} - x_r]\} \ \ \forall r = 1, 2, \cdots, n.$
Set $\forall r = 1, 2, \cdots, n \quad M_r(f) = \sup\{f(x) : x \in [x_{r-1} - x_r]\}$ and $\quad m_r(f) = \inf\{f(x) : x \in [x_{r-1} - x_r]\}$.
Now $\forall r = 1, 2, \cdots, n$, let us choose points $\xi_r, \eta_r \text{in} [x_{r-1}, x_r]$ such that $f(\xi_r) > M_r(f) - \frac{\epsilon}{5(b-a)}$ and $f(\eta_r) < m_r(f) + \frac{\epsilon}{5(b-a)}$.
Therefore $\forall r = 1, 2, \cdots, n$ we have
$M_r(f) - m_r(f) < f(\xi_r) + \frac{\epsilon}{5(b-a)} - f(\eta_r) + \frac{\epsilon}{5(b-a)}$
$\leq |f(\xi_r - f(\eta_r)| + \frac{2\epsilon}{5(b-a)}$
$\leq |f(\xi_r) - f_N(\xi_r)| + |f_N(\xi_r) - f_N(\eta_r) + |f_N(\eta_r) - f(\eta_r)| + \frac{2\epsilon}{5(b-a)}$
$< \frac{4\epsilon}{5(b-a)} + M_r(f_N) - m_r(f_N)$
Therefore
$\sum_{r=1}^n (M_r(f) - m_r(f))(x_r - x_{r-1}) \leq \frac{4\epsilon}{5(b-a)} \sum_{r=1}^n (x_r - x_{r-1}) + \sum_{r=1}^n (M_r(f_N) - m_r(f_N))(x_r - x_{r-1})$
or, $U(P, f) - L(P, f) \leq \frac{4\epsilon}{5} + \frac{\epsilon}{5} \quad by (ii)$
$= \epsilon$
Thus f is integrable on $[a,b]$.
Finally, to prove : $\int_a^b f(x)dx = \lim \int_a^b f_n(x)dx$.

Note that $\forall n$,
$|\int_a^b f(x)dx - \int_a^b f_n(x)dx| = |\int_a^b (f(x) - f_n(x))dx|$
$\leq \int_a^b |f(x) - f_n(x)|dx$
$\leq \int_a^b M_n dx$ where $M_n = \sup\{|f_n(x) - f(x)| : x \in [a,b]\}$.
$= M_n(b-a)$.
Since the sequence $\{f_n\}$ converges uniformly to f on $[a,b]$ we have $\lim M_n = 0$, by Theorem1.
Hence we conclude that $\int_a^b f(x)dx = \lim \int_a^b f_n(x)dx$.
Corollary: If $\{f_n\}$ is a sequence of continuous functions on $[a,b]$ converging uniformly to a function $f : [a,b] \to \mathbb{R}$ on $[a,b]$ then f and each f_n is integrable on $[a,b]$ and $\int_a^b f(x)dx = \lim \int_a^b f_n(x)dx$.
NOTE: The converse of Theorem 5 is not true.
Example47: Consider the sequence of functions $\{f_n\}$ on $[0,1]$ in Example 1. Recall that $\forall n$, $f_n : [0,1] \to \mathbb{R}$ is given by $f_n(x) = x^n \quad \forall x \in [0,1]$ and the limit function $f : [0,1] \to \mathbb{R}$ is given by

$$f(x) = \begin{cases} 1 & \text{for } 0 \leq x < 1 \\ 0 & \text{for } x = 1 \end{cases}$$

Each f_n being a continuous function on $[0,1]$ is integrable there.Again, f is abounded function having one one point of discontinuity viz. 1 in $[0,1]$; so, f is also integrable on $[0,1]$.
Clearly $\int_0^1 f(x)dx = 0$ and $\int_0^1 f_n(x)dx = \frac{1}{n+1}$.
Therefore, $\int_0^1 f(x)dx = \lim \int_0^1 f_n(x)dx$.
Finally, each f_n is continuous at $x = 1$ but the limit function is not continuous at $x = 1$. So by Theorem 3 it is evident that the convergence of the sequence $\{f_n\}$ to f is not uniform on $[0,1]$.[See note (i) of Ex. 19 also]. Thus the limit function of a sequence of Riemann integrable function may be Riemann integrable and the integral of the limit function may be equal to the limit of the integrals even if the convergence is not uniform.
NOTE: The converse of the Corollary of the Theorem 5 is not true.
Example48: Let us consider the sequence of functions $\{f_n\}$ on $[0,1]$ in Example 9. Note that $f_n(x) = \frac{Sinnx}{1+nx} \quad \forall x \in [0,1], \ \forall n$ and the limit function $f : [0,1] \to \mathbb{R}$ is given by $f(x) = 0 \quad \forall x \in [0,1]$.
Clearly f as well each f_n is continuous on $[0,1]$.
Now $\int_0^1 f(x)dx = 0$ and $\forall n, \quad |\int_0^1 f_n(x)dx| \leq \int_0^1 \frac{|Sinnx|}{1+nx}dx \leq \int_0^1 \frac{dx}{1+nx}$
But $\int_0^1 \frac{dx}{1+nx} = \frac{1}{n}\log(1+n) \to 0 as n \to \infty$.
Therefore $\lim \int_0^1 f_n(x)dx = 0$.
Hence $\int_0^1 f(x)dx = \lim \int_0^1 f_n(x)dx$. But $\{f_n\}$ does not converge uniformly on $[0,1]$. This fact have been proved in Example 43.
Thus the limit function of a sequence of continuous functions on $[a,b]$ may be continuous and the integral of the limit function may be equal to the limit of the integrals even if the convergence is

not uniform on $[a, b]$.

REMARK: Theorem 5 may help us to disprove uniform convergence of some sequences of functions on closed bounded intervals.

From this theorem we can immediately conclude that the sequence of functions $\{f_n\}$ in each of the examples 6,7 and 8 does not converge uniformly on $[0, 1]$.(See note (ii) & (iv) just after example 10).

REMARK: In theorem 3 we have seen that uniform convergence of a sequence of functions $\{f_n\}$ is sufficient to transit continuity from individual term to the limit function. That uniform convergence is not necessary for such transition has been supported by Example 41. Moreover, for a sequence of continuous functions $\{f_n\}$ on $[0, 1]$ with continuous limit function f neither

(i) $x \in [0, 1], \{x_n\}$ in $[0, 1]$ with $x_n \to x \Longrightarrow f_n(x) \to f(x)$ (See Example 42)

nor (ii) $\int_0^1 f(x)dx = \lim \int_0^1 f_n(x)dx$ (See Example 48)

can ensure uniform convergence of the sequence $\{f_n\}$ on $[0, 1]$. In Example 42,$f_n(x) = \frac{nx}{1+n^2x^2} \quad \forall x \in [0, 1], \forall n; f(x) = 0 \quad \forall x \in [0, 1]$ and $\int_0^1 f_n(x)dx = \frac{1}{2n} \int_0^1 \frac{2n^2x}{1+n^2x^2} dx = \frac{1}{2n} tan^{-1}(n^2) \to 0 as n \to \infty$ so that $\int_0^1 f(x)dx = 0 = \lim \int_0^1 f_n(x)dx$.

Thus for a sequence $\{f_n\}$ of continuous functions on $[0, 1]$ with continuous limit function the conditions (i) and (ii) above, even jointly cannot guarantee the uniform convergence of the sequence $\{f_n\}$.

Now for a sequence of the above type (i.e. a sequence of continuous functions on $[0, 1]$ with continuous limit function) let us proceed to investigate the extra condition that would make the sequence uniformly convergent on $[0, 1]$.

First of all let us have a closer look to Example 32. In this example the sequence $\{x^n(1 - x)\}$ of continuous functions on $[0, 1]$ converges uniformly to the zero function on $[0, 1]$. Here it is to be noted that $x(1 - x) \geq x^2(1 - x) \geq x^3(1 - x) \geq ...$ for any x in $[0, 1]$.

It immediately comes in mind that if $\{g_n\}$ is a sequence of continuous functions on $[0, 1]$ with zero as its limit function and $g_1(x) \geq g_2(x) \geq g_3(x) \geq \cdots \quad \forall x \in [0, 1]$ then the convergence of the sequence of functions $\{g_n\}$ would probably be uniform on $[0, 1]$. Really, this is true. Let us explain.

If possible, we suppose that the convergence of $\{g_n\}$ is not uniform on $[0, 1]$. So there exists $\epsilon_0 > 0$ such that for no positive integer N the following is true:

$g_n(x) < \epsilon_0 \quad \forall x \in [0, 1], \ \forall n \geq N$ (as $g_n(x) \geq 0 \ \forall x \in [0, 1]$).

For such an $\epsilon_0 > 0$, we choose for each $n \in N$ a point x_n in $[0, 1]$ such that $g_n(x_n) \geq \epsilon_0$.

Since $\{x_n\}$ is a sequence in the bounded set $[0, 1]$ it has a cluster point $x_0 \in \mathbb{R}$ (By Bolzano-Weierstrass Theorem). Again since $[0, 1]$ is a closed set so $x_0 \in [0, 1]$. Let us choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim x_{n_k} = x_0$.

Fix $m \in \mathbb{N}$. For sufficiently large k, $n_k > m$ and for such k, $g_m(n_k) \geq g_{n_k}(x_{n_k}) \geq \epsilon_0$.

Now, by continuity of g_m at x_0 $\lim g_m(x_{n_k}) = g_m(x_0)$.

Therefore, we have $g_m(x_0) \geq \epsilon_0$.

Since $m \in \mathbb{N}$ is arbitrary it follows that $g_m(x_0) \geq \epsilon_0 \ \ \forall m \in \mathbb{N}$. But $\lim g_m(x_0) = 0$.

Therefore we have $0 \geq \epsilon_0 \rightarrow$ a contradiction.

Thus actually we have proved the following:

THEOREM6: Let $\{g_n\}$ be a sequence of continuous functions on $[0,1]$ with the zero as its limit function. If $g_1(x) \geq g_2(x) \geq g_3(x) \geq \cdots \quad \forall x \in [0,1]$ then the convergence of the sequence of functions $\{g_n\}$ is uniform on $[0,1]$.

EXAMPLE49: The sequence of functions $\{x^n(1-x)\}$ in Example 32 satisfies all the conditions of the above theorem and hence converges uniformly on $[0,1]$.

EXAMPLE50: Consider the sequence of functions $\{f_n\}$ of example 33. Recall that for a given continuous function ψ on $[0,1]$ with $\psi(1) = 0, f_n(x) = x^n\psi(x) \ \ \forall x \in [0,1], \ \forall n$.

Set $g_n(x) = |f_n(x)| \ \ \forall x \in [0,1]$.

Clearly $\{g_n\}$ satisfies all the condition of Theorem 6 and so $\{g_n\}$ converges uniformly to 0 on $[0,1]$. Hence $\{f_n\}$ also converges uniformly to 0 on $[0,1]$.

Corollary: (i) Let $\{g_n\}$ be a sequence of continuous functions on $[0,1]$ with as its limit function. If $g_1(x) \leq g_2(x) \leq \cdots \quad \forall x \in [0,1]$ then the convergence is uniform in $[0,1]$.

Corollary: (ii) Let $\{f_n\}$ be a sequence of continuous functions on $[0,1]$ having a continuous limit function. If $f_1(x) \geq f_2(x) \geq f_3(x) \geq \cdots \quad \forall x \in [0,1]$ or if $f_1(x) \leq f_2(x) \leq f_3(x) \leq \ldots \quad \forall x \in [0,1]$ then the convergence of the sequence of functions $\{f_n\}$ is uniform on $[0,1]$.

Problem7: Prove uniform convergence of the sequence of functions of Example 17 and 31 using corollary (ii) above.

REMARK : The proof of Theorem 6 depends on the fact that the interval $[0,1]$ is bounded and closed subset of $\mathbb{R}$. So, question may arise:

How far can we generalize the Theorem 6 ?

To answer this we have the following theorem :

THEOREM7:(Known as Dini's theorem on uniform convergence)

Let S be a non-empty closed bounded subset of $\mathbb{R}$ and $\{f_n\}$ be a sequence of continuous functions on S converging to a continuous function f on S. If $f_1(x) \geq f_2(x) \geq f_3(x) \geq \cdots \quad \forall x \in S$ or if $f_1(x) \leq f_2(x) \leq f_3(x) \leq \cdots \quad \forall x \in S$ then the convergence of the sequence of functions $\{f_n\}$ to f is uniform on S.

Proof is similar to the proof of Theorem 6.

NOTE 1: Boundedness of S in the above theorem is essential.

EXAMPLE51: Consider the sequence of functions $\{h_n\}$ on $\{x : x \geq 0\}$ in Example 15. Recall that $h_n(x) = \frac{n}{x+n} \quad \forall x \geq 0, \ \forall n$. Clearly each h_n is continuous on $\{x : x \geq 0\}$ and the limit function $h : \{x : x \geq 0\} \to \mathbb{R}$ defined by $h(x) = 1 \ \forall x \geq 0$ is obviously continuous on $\{x : x \geq 0\}$.

Now $\forall n \in N$ and $\forall x/ge0$ we have

$h_n(x) - h_{n+1}(x) = \frac{n}{x+n} - \frac{n+1}{x+n+1} \leq 0$.

In example 15 we have proved that $\{h_n\}$ does not converge uniformly on $\{x : x \geq 0\}$.

Note that $\{x : x \geq 0\}$ is a closed but unbounded subset of $\mathbb{R}$.This suggests that boundedness of S in Theorem 7 is essential.

NOTE: (ii) Closedness of S in Theorem 7 is essential.

EXAMPLE52: Consider the sequence of functions $\{f_n\}$ on $(0, 1]$ in Example 16. Recall that $f_n(x) = \frac{1}{nx+1} \quad \forall x \in (0, 1], \ \forall n$. Clearly each f_n is continuous on $(0, 1]$ and the limit function $f : (0, 1] \to \mathbb{R}$ defined by $f(x) = 0 \ \forall x \in (0, 1]$ is obviously continuous on $(0, 1]$.

Now $\forall n \in N$ and $\forall x \in (0, 1]$ we have

$f_n(x) - f_{n+1}(x) = \frac{1}{nx+1} - \frac{1}{(n+1)x+1} > 0$.

In example 16 we have proved that $\{f_n\}$ does not converge uniformly on $(0, 1]$.

Note that $(0, 1]$ is bounded but not a closed subset of $\mathbb{R}$.This suggests that closedness of S in Theorem 7 is essential.

NOTE: (iii) For uniform convergence of a sequence $\{f_n\}$ of continuous functions on a closed bounded subset S of $\mathbb{R}$ neither

$f_1(x) \geq f_2(x) \geq f_3(x) \geq \cdots \quad \forall x \in S$

nor $f_1(x) \leq f_2(x) \leq f_3(x) \leq \cdots \quad \forall x \in S$

is essential.

EXAMPLE53: Consider the sequence of functions $\{f_n\}$ on $[-\frac{1}{2}, \frac{1}{2}]$ defined by $f_n(x) = x|x|^n \quad \forall x \in [-\frac{1}{2}, \frac{1}{2}], \ \forall n$.

Obviously the limit function $f : [-\frac{1}{2}, \frac{1}{2}] \to \mathbb{R}$ is given by $f(x) = 0 \quad \forall x \in [-\frac{1}{2}, \frac{1}{2}]$.(Converse of Theorem 7 is not true).Note that $\text{Sup}\{|f_n(x) - f(x)| : x \in [-\frac{1}{2}, \frac{1}{2}]\} = (\frac{1}{2})^{n+1} \to 0$ as $n \to \infty$.

The sequence $\{f_n\}$ converges uniformly to f on $[-\frac{1}{2}, \frac{1}{2}]$.

Again $f_1(x) \geq f_2(x) \geq f_3(x) \geq \cdots \quad \forall x \in [0, \frac{1}{2}]$ and $f_1(x) \leq f_2(x) \leq f_3(x) \leq \cdots \quad \forall x \in [-\frac{1}{2}, 0)$.

finally note that $\{f_n\}$ is a uniformly convergent sequence of continuous functions on the closed bounded subset $[-\frac{1}{2}, \frac{1}{2}]$ of $\mathbb{R}$ without satisfying the condition of monotonicity on $\{f_n\}$.

Uniform convergence and Differentiability :

By analogy with Theorems 3 and 5 one might expect that if a sequence of functions $\{f_n\}$ converges uniformly to f on $[a, b]$ and each f_n is differentiable at some point c in $[a, b]$ then f is also differentiable at c and moreover $\lim f_n'(c) = f'(c)$. But this is not true.

EXAMPLE54: Let us consider a sequence of polynomials $\{P_n(x)\}$ converging uniformly to the continuous function $|x|$ on $[-1,1]$; such choice of $\{P_n\}$ is possible by weierstrass approximation theorem which states that "any continuous function on a closed bounded interval can be uniformly approximated by polynomials".

Now,each P_n is differentiable at $x = 0$ but $|x|$ is not differentiable at $x = 0$. So, it is clear that uniform convergence does not preserve differentiability.

EXAMPLE55: Let us consider the sequence of functions on $[-1,1]$ defined by $f_n(x) = \frac{Sinnx}{\sqrt{n}} \quad \forall x \in [-1,1], \ \forall n$.

Note that $|f_n(x)| \leq \frac{1}{\sqrt{n}} \quad \forall x \in [-1,1], \ \forall n$ so that $\lim |f_n(x)| = 0 \quad \forall x \in [-1,1]$ and hence the limit function f is given by $f(x) = \lim f_n(x) = 0 \quad \forall x \in [-1,1]$.

Obviously $\text{Sup}\{|f_n(x) - f(x)| : x \in [-1,1]\} \leq \frac{1}{\sqrt{n}} \to 0$ so that the convergence of $\{f_n\}$ to f is uniform on $[-1,1]$.(This fact also follows from example 12 and problem 3).

Now $f_n'(0) = \sqrt{n} \ \forall n$ and $f'(0) = 0$.

Therefore $\lim f_n'(0) \neq f(0)$, since $f_n'(0) = \infty$.

So even if the uniform limit f of a sequence of functions $\{f_n\}$ each f_n being differentiable at some point c is also differentiable at c, the equality $\lim f_n'(c) = f(c)$ need not be true.

Problem8: Cite an example of a sequence of functions $\{f_n\}$ converging uniformly to a function f on an interval $[a,b]$ such that f as well as each f_n is differentiable at some point c in $[a,b]$ but $\lim f_n'(c) = f(c)$ although $\lim f_n'(c)$ is finite.

THEOREM8: Suppose that $\{f_n\}$ is a continuously differentiable functions on $[a,b]$ converging to a function $f : [a,b] \to \mathbb{R}$. If the sequence of functions $\{f_n'\}$ converges uniformly on $[a,b]$ then f is differentiable on $[a,b]$ and $\lim f_n'(x) = f(x) \quad \forall x \in [a,b]$.

PROOF: Let $F : [a,b] \to \mathbb{R}$ be the uniform limit of the sequence of functions $\{f_n'\}$ on $[a,b]$. Since each f_n' is continuous on $[a,b]$, by Theorem 3 it follows that F is also continuous on $[a,b]$.

Again, by Corollary of theorem 5, for each $x \in [a,b]$ we have $\int_a^x F(t)dt = \lim \int_a^x f_n'(t)dt$.

So, by Fundamental theorem of Calculus ,

$\forall x \in [a,b], \ \forall n$ we have $\int_a^x f_n'(t)dt = f_n(x) - f_n(a)$.

Therefore, $\forall x \in [a,b] \quad \int_a^x F(t)dt = \lim(f_n(x) - f_n(a)) = \lim f_n(x) - \lim f_n(a) = f(x) - f(a) \qquad (*)$.

Finally, since F is continuous on $[a,b]$, $\int_a^x F(t)dt$ is differentiable on $[a,b]$ and hence from $(*)$ $f(x)$ is differentiable on $[a,b]$ with $f'(x) = \frac{d}{dx}\int_a^x F(t)dt$.

Therefore $\forall x \in [a,b], \quad f'(x) = F(x) = \lim f_n'(x)$.

REMARK: If we do not assume the continuity of the functions f_n' on $[a,b]$ in the above Theorem, then stronger hypothesis are required for the assertion $f_n' \to f'$ if $f_n \to f$.

THEOREM9: Suppose that $\{f_n\}$ is a continuously differentiable functions on $[a,b]$ such that

$\{f_n(x_0)\}$ converges for some point x_0 in $[a,b]$.If $\{f_n'\}$ converges uniformly on $[a,b]$, then $\{f_n\}$ converges uniformly on $[a,b]$ to a function f and moreover, f is differentiable on $[a,b]$ satisfying $f'(x) = \lim f_n'(x) \quad \forall x \in [a,b]$.

PROOF: Suppose that $\{f_n'\}$ converges uniformly on $[a,b]$.

To prove : $\{f_n\}$ converges uniformly on $[a,b]$.

Let $\epsilon > 0$ be a given real.

By Uniform convergence of the sequence of functions $\{f_n'\}$ on $[a,b]$ and by convergence of the real sequence $\{f_n(x_0)\}$ let us choose a positive integer N such that

(i) $|f_m'(x) - f_n'(x)| < \frac{\epsilon}{2(b-a)} \quad \forall x \in [a,b], \quad \forall m,n \geq N$

and (ii) $|f_m(x_0) - f_n(x_0)| < \frac{\epsilon}{2} \qquad \forall m,n \geq N$.

For x,t in $[a,b]$ applying Mean Value Theorem on $f_m - f_n \quad m,n \geq N$ we have

$(f_m(x) - f_n(x)) - (f_m(t) - f_n(t)) = (x-t)(f_m'(\xi_{m,n}) - f_n'(\xi_{m,n}))$ for some point $\xi_{m,n}$ in between x and t.

Therefore for x,t in $[a,b]$, $\quad \forall m,n \geq N$ we have by (i)

(iii) $|f_m(x) - f_n(x) - f_m(t) + f_n(t)| \leq |x-t|.\frac{\epsilon}{2(b-a)}$

$|f_m(x) - f_n(x) - f_m(t) + f_n(t)| \leq \frac{\epsilon}{2} \quad \forall x,t \in [a,b], \ \forall m,n \geq N$ (iv)

Therefore $\forall x \in [a,b], \quad \forall m,n \geq N$ we get

$|f_m(x)-f_n(x)| \leq |f_m(x)-f_n(x)-f_m(x_0)+f_n(x_0)|+|f_m(x_0)-f_n(x_0)| < \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon$ by (ii) and (iv).

So by Cauchy criterion for uniform convergence (Theorem 2) we conclude that the sequence of functions $\{f_n\}$ converges uniformly on $[a,b]$.

Next suppose that f is the uniform limit of $\{f_n\}$ on $[a,b]$. We have to prove that : f is differentiable on $[a,b]$ and (v) $f'(x) = \lim f_n'(x) \quad \forall x \in [a,b]$.

Given : $\{f_n'\}$ converges uniformly on $[a,b]$.

Let F be the uniform limit of $\{f_n'\}$ on $[a,b]$.

Therefore (v) is same as (vi): $f'(x) = F(x) \quad \forall x \in [a,b]$.

Fix $x \in [a,b]$. Set $\forall t \in [a,b] - \{x\}$

$$\phi(t) = \frac{f(t) - f(x)}{t - x} \quad \text{and} \quad \phi_n(t) = \frac{f_n(t) - f_n(x)}{t - x} \ \forall n.$$

Therefore

$$\lim_{t \to x} \phi_(t) = f'(x), \text{if it exists}, \text{(vii)} \lim_{t \to x} \phi_n(t) = f_n'(t) \quad \forall n, \text{(viii)} \lim_{n \to \infty} \phi_n(t) = \phi(t) \quad \forall t \in [a,b] - \{x\}$$

Note that $\forall t \in [a,b] - \{x\}$

$|\phi_m(t) - \phi_n(t)| < \frac{\epsilon}{2(b-a)} \quad \forall m,n \geq N$ using (iii).

Therefore $\{\phi_n\}$ converges uniformly on $[a,b] - \{x\}$ (by Theorem 2) and (iii) shows that the uniform limit of $\{\phi_n\}$ is ϕ on $[a,b]-\{x\}$. Finally (vi) and the defintion of ϕ suggests that at the fixed $x \in [a,b]$, we have to establish that

$$\lim_{t\to x} \phi(t) = F(x).$$

Note that $\forall t \in [a,b]-\{x\}, \quad \forall n \in \mathbb{N}$ we have

$|\phi(t) - F(x)| \leq |\phi(t) - \phi_n(t)| + |\phi_n(t) - f'_n(x)| + |f'_n(x) - F(x)|$ ....(ix).

Let $\epsilon > 0$ be a given real.

Since ϕ is the uniform limit of $\{\phi_n\}$ on $[a,b]-\{x\}$, $\lim_{n\to\infty} f'_n(x) = F(x)$ we can choose a positive integer N_0 Such that $\forall n \geq N_0$,

$$(x)\begin{cases} |\phi_n(t) - \phi(t)| < \frac{\epsilon}{3} & \forall t \in [a,b]-\{x\} \\ |\phi'_n(x) - F(x)| < \frac{\epsilon}{3} \end{cases}$$

Now for a fixed $n \geq N_0$, by (vii) we choose a $\delta > 0$ such that $\forall t \in (x-\delta, x+\delta) \cap ([a,b]-\{x\})$ we have

$$|\phi_n(t) - f'_n(x)| < \frac{\epsilon}{3} \qquad ....(xi)$$

Hence from $(ix),(x)$and(xi) we get

$|\phi(t) - F(x)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \qquad \forall t \in (x-\delta, x+\delta) \cap ([a,b]-\{x\})$

Therefore $\lim_{t\to x} \phi(t)$ exists and is equal to $F(x)$.

Therefore $f'(x) = F(x)$.

Since $x \in [a,b]$ is arbitrary it follows that f is differentiable on $[a,b]$ and $f'(x) = F(x) = \lim_{n\to\infty} f'_n(x) \quad \forall x \in [a,b]$.

EXAMPLE56: To verify Theorem 8 for the sequence of functions $\{f_n\}$ on $[0,\frac{1}{2}]$ defined by $f_n(x) = 1 + x + x^2 + \cdots + x^{n-1} \quad \forall x \in [0,\frac{1}{2}], \ \forall n$.

Note that,for each n, f_n being a polynomial function on $[0,\frac{1}{2}]$ is obviously continuously differentiable on $[0,\frac{1}{2}]$.

Now, $f_n(x) = 1 + x + x^2 + \cdots + x^{n-1} = \frac{1-x^n}{1-x} \quad \forall x \in [0,\frac{1}{2}], \forall n$.

Since $\forall x \in [0,\frac{1}{2}]$, the sequence $\{x^n\}$ converges to zero, it follows that the limit function f of $\{f_n\}$ is given by $f(x) = \frac{1}{1-x} \quad \forall x \in [0,\frac{1}{2}]$.

Clearly f is differentiable on $[0,\frac{1}{2}]$ and $f'(x) = \frac{1}{(1-x)^2} \quad \forall x \in [0,\frac{1}{2}]$.

The verification of the Theorem 8 will be complete if we prove that f' is the uniform limit of the sequence of functions $\{f'_n\}$ on $[0,\frac{1}{2}]$.

Now $f'_1(x) = 0 \quad \forall x \in [0,\frac{1}{2}]$ and $\forall n \geq 2, f'_n(x) = 1 + 2x + 3x^2 + \cdots + (n-1)x^{n-2} \quad \forall x \in [0,\frac{1}{2}]$.

Therefore for $n \geq 2$, for any x in $[0,\frac{1}{2}]$ we have

$$\begin{aligned}
|f_n'(x) - f'(x)| &= \left|1 + 2x + 3x^2 + \cdots + (n-1)x^{n-2} - \frac{1}{(1-x)^2}\right| \\
&= \frac{1}{(1-x)}\left|\{1 + 2x + 3x^2 + \cdots + (n-1)x^{n-2}\}.(1-x) - \frac{1}{1-x}\right| \\
&= \frac{1}{(1-x)}\left|1 + x + x^2 + \cdots + x^{n-1} - \frac{1}{1-x}\right| \\
&= \frac{1}{(1-x)}\left|\frac{1-x^n}{1-x} - \frac{1}{1-x}\right| \leq \frac{x^n}{(1-x)^2}
\end{aligned}$$

. Therefore $\forall n \geq 2, \text{Sup}\{|f_n'(x) - f'(x)| : x \in [0, \frac{1}{2}]\} \leq \frac{1}{2^{n-2}}$ $[\because 0 \leq x \leq \frac{1}{2}]$.
Since $\frac{1}{2^{n-2}}$ converges to zero it follows that

$$\lim_{n\to\infty} \text{Sup}\{|f_n'(x) - f'(x)| : x \in [0, \frac{1}{2}]\} = 0.$$

Thus $\{f_n'\}$ converges uniformly to f' on $[0, \frac{1}{2}]$.

Problem9: Verify Theorem 9 for the sequence of functions $\{f_n\}$ of the above example 56.

Problem10: Suppose that the sequence of functions $\{f_n\}$ on $[0,1]$ is defined by $f_n(x) = \frac{n^2x^2}{1+n^2x^2}$ $\forall x \in [0,1]$, $\forall n$. Show that $\{f_n\}$ does not satisfy all the conditions of Theorem 9 but the derivative of the limit function exists in $[0,1]$ and is equal to the limit of the derivatives at any point $[0,1]$.

Problem11: Suppose that the sequence of functions $\{f_n\}$ on $[-1,1]$ is defined by $f_n(x) = \frac{x}{1+nx^2}$ $\forall x \in [-1,1]$, $\forall n$. Show that $\{f_n\}$ converges uniformly to a function f on $[-1,1]$, and that the equation $f'(x) = \lim_{n\to\infty} f_n'(x)$ is correct if x is a point in $[-1,1] - \{0\}$, but false if $x = 0$.