

47. Conservative Vector Fields

Given a function $z = \phi(x, y)$, its gradient is $\nabla\phi = \langle \phi_x, \phi_y \rangle$. Thus, $\nabla\phi$ is a **gradient** (or **conservative**) **vector field**, and the function ϕ is called a **potential function**.

Suppose we are given the vector field first, in the form $\mathbf{F}(x, y) = \langle M(x, y), N(x, y) \rangle$. Can we show that this is a conservative vector field? Recall that $\phi_{xy} = \phi_{yx}$ is true by Clairaut's Theorem. Assuming such a function ϕ exists, we infer that $\phi_x = M$ and that $\phi_y = N$, and observing that $\phi_{xy} = \phi_{yx}$, this is equivalent to showing that $M_y = N_x$. In other words, if ϕ exists, then $M_y = N_x$, and if $M_y = N_x$ is true, then ϕ exists. (The exceptions to this property are rare and not relevant to this discussion).

To summarize, if given a vector field $\mathbf{F}(x, y) = \langle M(x, y), N(x, y) \rangle$, then two cases result:

- If $M_y = N_x$, then \mathbf{F} is conservative, and there exists a potential function ϕ .
- If $M_y \neq N_x$, then \mathbf{F} is not conservative and no such potential function ϕ exists.



Example 47.1: Determine if $\mathbf{F}(x, y) = \langle 3x^2y^2, 2x^3y \rangle$ is conservative. If it is, find a potential function $\phi(x, y)$ such that $\nabla\phi = \mathbf{F}$.

Solution: From \mathbf{F} , we have $M(x, y) = 3x^2y^2$ and $N(x, y) = 2x^3y$. We find M_y and N_x :

$$M_y = 6x^2y \quad \text{and} \quad N_x = 6x^2y.$$

Since $M_y = N_x$, then \mathbf{F} is conservative, and there exists a function $\phi(x, y)$ such that $\phi_x = 3x^2y^2$ and $\phi_y = 2x^3y$. Since we assume that $\phi_x = 3x^2y^2$, we integrate it with respect to x :

$$\int 3x^2y^2 \, dx = x^3y^2 + g(y).$$

Here, $g(y)$ represents a possible term in variable y , noting that under differentiation with respect to x , $\frac{\partial}{\partial x}g(y) = 0$. Now differentiate this result with respect to y :

$$\frac{\partial}{\partial y}(x^3y^2 + g(y)) = 2x^3y + g'(y).$$

This is compared to $N(x, y) = 2x^3y$:

$$2x^3y + g'(y) = 2x^3y.$$

This suggests $g'(y) = 0$ so that integrating, $g(y) = k$, a constant. Thus, the potential function has the form $\phi(x, y) = x^3y^2 + k$. In such cases, any constants are set to 0. This leaves

$$\phi(x, y) = x^3y^2$$

as a potential function of \mathbf{F} . This is easily checked by showing that $\phi_x = 3x^2y^2$ and $\phi_y = 2x^3y$.



Example 47.2: Determine if $\mathbf{F}(x, y) = \langle xy, 1 - x^2 \rangle$ is conservative. If it is, find the potential function $\phi(x, y)$ such that $\nabla\phi = \mathbf{F}$.

Solution: We have $M(x, y) = xy$ and $N(x, y) = 1 - x^2$. Observe that $M_y = x$ and $N_x = -2x$. Since $M_y \neq N_x$, vector field \mathbf{F} is not conservative, and there does not exist a function whose gradient is \mathbf{F} .



Example 47.3: Determine if $\mathbf{F}(x, y) = \langle y - 3, x + 2 \rangle$ is conservative. If it is, find a potential function ϕ .

Solution: We have $M(x, y) = y - 3$ and $N(x, y) = x + 2$. Observe that $M_y = 1$ and $N_x = 1$. Since $M_y = N_x$, the vector field \mathbf{F} is conservative. To determine $\phi(x, y)$, we first integrate $M(x, y)$ with respect to x :

$$\int (y - 3) dx = xy - 3x + g(y).$$

Differentiating this result with respect to y , we have

$$\frac{\partial}{\partial y}(xy - 3x + g(y)) = x + g'(y).$$

This is compared to $N(x, y) = x + 2$:

$$x + g'(y) = x + 2.$$

Thus, $g'(y) = 2$, so that $g(y) = 2y + k$ (any constants of integration can be set to 0). A potential function is

$$\phi(x, y) = xy - 3x + 2y,$$

which we check by showing that $\nabla\phi(x, y) = \mathbf{F}(x, y)$:

$$\phi_x(x, y) = y - 3, \quad \phi_y(x, y) = x + 2.$$

These are precisely the components of \mathbf{F} , so $\nabla\phi(x, y) = \mathbf{F}(x, y)$.



Given a conservative vector field $\mathbf{F}(x, y) = \langle M(x, y), N(x, y) \rangle$, a “shortcut” to find a potential function $\phi(x, y)$ is to integrate $M(x, y)$ with respect to x , and $N(x, y)$ with respect to y , and to form the union of the terms in each antiderivative. However, check that the alleged potential function’s partial derivatives with respect to x , and respect to y , do give M and N , respectively.

Example 47.4: Given the conservative vector field $\mathbf{F}(x, y) = \langle 3x^2y^2, 2x^3y \rangle$, find a potential function, $\phi(x, y)$.

Solution: Integrate $M(x, y) = 3x^2y^2$ with respect to x , and $N(x, y) = 2x^3y$ with respect to y :

$$\int 3x^2y^2 dx = x^3y^2 \quad \text{and} \quad \int 2x^3y dy = x^3y^2.$$

Observing the two antiderivatives, we infer that $\phi(x, y) = x^3y^2$ may be a potential function. A check that $\phi_x = M(x, y) = 3x^2y^2$ and $\phi_y = N(x, y) = 2x^3y$ shows that this function is a correct potential function. Any constants of integration can be ignored.



Example 47.5: Given the conservative vector field $\mathbf{F}(x, y) = \langle 3x^2 + 2y, 2x - 2y \rangle$, find the potential function, $\phi(x, y)$.

Solution: Integrate $3x^2 + 2y$ with respect to x , and $2x - 2y$ with respect to y :

$$\int (3x^2 + 2y) dx = x^3 + 2xy \quad \text{and} \quad \int (2x - 2y) dy = 2xy - y^2.$$

The union of terms from these two antiderivatives is $\phi(x, y) = x^3 + 2xy - y^2$. We check that this is actually a potential function: $\phi_x = 3x^2 + 2y = M(x, y)$, and $\phi_y = 2x - 2y = N(x, y)$. Thus, this is a correct potential function.



Example 47.6: A student is given the vector field $\mathbf{F}(x, y) = \langle x^2, xy \rangle$. He then integrates x^2 with respect to x , getting $\int x^2 dx = \frac{1}{3}x^3$, and integrates xy with respect to y , getting $\int xy dy = \frac{1}{2}xy^2$. He concludes that the potential function is $\phi(x, y) = \frac{1}{3}x^3 + \frac{1}{2}xy^2$. Explain the error.

Solution: Vector field \mathbf{F} is *not* conservative since $M_y \neq N_x$. Thus, there is no potential function that generates \mathbf{F} . Note that the alleged potential function, $\phi(x, y) = \frac{1}{3}x^3 + \frac{1}{2}xy^2$, does not generate \mathbf{F} since $\phi_x = x^2 + \frac{1}{2}y^2$, which is not equal to x^2 . In other words, $\nabla\phi \neq \mathbf{F}$.



Vector fields in R^3 can also be conservative, where $w = \phi(x, y, z)$ is a potential function of a vector field $\nabla\phi = \mathbf{F}(x, y, z) = \langle \phi_x, \phi_y, \phi_z \rangle$. However, showing that a vector field \mathbf{F} in R^3 is conservative is found by showing that $\text{curl } \mathbf{F} = \mathbf{0}$. The **curl** of a vector field is discussed in Section 52.

48. Fundamental Theorem of Line Integrals

If \mathbf{F} is a conservative vector field in R^2 with $\phi(x, y)$ as its potential function, and C is a directed path with endpoints $a = (x_0, y_0)$ and $b = (x_1, y_1)$, then

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= [\phi(x, y)]_a^b \\ &= \phi(b) - \phi(a) \\ &= \phi(x_1, y_1) - \phi(x_0, y_0).\end{aligned}$$

This is called the **Fundamental Theorem of Line Integrals** (FTLI). In this case, there is no need to parametrize the path, as the value of the line integral depends only on the potential function evaluated at the endpoints, then subtracted in the usual manner of integration.

A couple of corollaries follow:

- Line integrals in a conservative vector field are *path independent*, meaning that any path from a to b will result in the same value of the line integral.
- If the path C is a simple loop, meaning it starts and ends at the same point and does not cross itself, and \mathbf{F} is a conservative vector field, then the line integral is 0.



Example 48.1: Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = \langle 3x^2y^2, 2x^3y \rangle$ and C is the line segment from $a = (1, 2)$ to $b = (4, -3)$.

Solution: From a previous example, we showed that \mathbf{F} is conservative, and that a potential function is $\phi(x, y) = x^3y^2$. Therefore,

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= [x^3y^2]_{(1,2)}^{(4,-3)} \\ &= (4)^3(-3)^2 - (1)^3(2)^2 \\ &= 576 - 4 = 572.\end{aligned}$$

Note that we did not actually parametrize the line segment to solve this line integral.



Example 48.2: Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = \langle 2x, 3y \rangle$ and C is any path from $a = (1, 0)$ to $b = (0, 1)$.

Solution: Let's try a few common paths. Suppose C is a line from a to b . We have $\mathbf{r}(t) = \langle 1 - t, t \rangle$, where $0 \leq t \leq 1$. Thus, $\mathbf{r}'(t) = \langle -1, 1 \rangle$ and $\mathbf{F}(t) = \langle 2(1 - t), 3t \rangle$. The line integral is

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \langle 2(1 - t), 3t \rangle \cdot \langle -1, 1 \rangle dt \\ &= \int_0^1 (5t - 2) dt \\ &= \left[\frac{5}{2}t^2 - 2t \right]_0^1 \\ &= \frac{5}{2} - 2 \\ &= \frac{1}{2}. \end{aligned}$$

Now suppose C is a quarter circle, centered at the origin, with radius 1. It is parametrized as $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$, where $0 \leq t \leq \frac{\pi}{2}$. As a result, $\mathbf{r}'(t) = \langle -\sin t, \cos t \rangle$ and $\mathbf{F}(t) = \langle 2 \cos t, 3 \sin t \rangle$. The line integral is

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{\pi/2} \langle 2 \cos t, 3 \sin t \rangle \cdot \langle -\sin t, \cos t \rangle dt \\ &= \int_0^{\pi/2} \sin t \cos t dt \\ &= \left[\frac{1}{2} \sin^2 t \right]_0^{\pi/2} \quad \left\{ \begin{array}{l} \text{Letting } u = \sin t \\ \text{so that } du = \cos t dt. \end{array} \right. \\ &= \frac{1}{2}. \end{aligned}$$

Suppose C is a parabola $x = 1 - y^2$. It is parametrized as $\mathbf{r}(t) = \langle 1 - t^2, t \rangle$, where $0 \leq t \leq 1$. Thus, $\mathbf{r}'(t) = \langle -2t, 1 \rangle$ and $\mathbf{F}(t) = \langle 2(1 - t^2), 3t \rangle$. The line integral is

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \langle 2(1 - t^2), 3t \rangle \cdot \langle -2t, 1 \rangle dt \\ &= \int_0^1 (4t^3 - t) dt \\ &= \left[t^4 - \frac{1}{2}t^2 \right]_0^1 \\ &= 1 - \frac{1}{2} \\ &= \frac{1}{2}.\end{aligned}$$

It appears that regardless the path, the line integral is $\int_C \mathbf{F} \cdot d\mathbf{r} = \frac{1}{2}$. Although three examples are not a “proof” that this assertion is true, it suggests that it might be worth approaching the problem from a different perspective.

Observe that the vector field \mathbf{F} is conservative: $M(x, y) = 2x$, so that $M_y = 0$, and $N(x, y) = 3y$, so that $N_x = 0$. A potential function is $\phi(x, y) = x^2 + \frac{3}{2}y^2$ (You should verify this). Thus, using the Fundamental Theorem of Line Integrals, we have

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \left[x^2 + \frac{3}{2}y^2 \right]_{(1,0)}^{(0,1)} \\ &= \left((0)^2 + \frac{3}{2}(1)^2 \right) - \left((1)^2 + \frac{3}{2}(0)^2 \right) \\ &= \frac{3}{2} - 1 \\ &= \frac{1}{2}.\end{aligned}$$

This example illustrates that in a conservative vector field, the line integral along any path between two fixed endpoints will always give the same result. Rather than try many different paths, it's easier to first check whether \mathbf{F} is conservative. If it is, then skip the parametrization step entirely, and proceed to finding a potential function and using the Fundamental Theorem of Line Integrals.



Example 48.3: Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = \langle y, x + 2y \rangle$ and C is a sequence of line segments from $(1, 3)$ to $(2, 7)$ to $(-4, 0)$ to $(8, 2)$.

Solution: We check first to see if \mathbf{F} is conservative: $M_y = 1$ and $N_x = 1$. Since $M_y = N_x$, then \mathbf{F} is conservative, and it is not necessary to parametrize the sequence of line segments. Instead, we find ϕ and evaluate it by using the Fundamental Theorem of Line Integrals. We need a potential function. Note that

$$\int y \, dx = xy \quad \text{and} \quad \int (x + 2y) \, dy = xy + y^2.$$

Thus, $\phi(x, y) = xy + y^2$ is the (probable) potential function. We check by finding $\nabla\phi$: $\phi_x = y$ and $\phi_y = x + 2y$. These are M and N , respectively, so $\phi(x, y) = xy + y^2$ is a correct potential function. Therefore,

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= [xy + y^2]_{(1,3)}^{(8,2)} \\ &= ((8)(2) + (2)^2) - ((1)(3) + (3)^2) \\ &= 20 - 12 \\ &= 8. \end{aligned}$$

All of the intermediate points were ignored. We only needed the starting and ending point of the path.



Example 48.4: Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = \langle 2x, 3y^2 \rangle$ and C is given by $\mathbf{r}(t) = \langle t^2, 5t \rangle$ for $-1 \leq t \leq 3$.

Solution: Note that $M_y = 0$ and that $N_x = 0$. Since $M_y = N_x$, then \mathbf{F} is conservative, and that $\phi(x, y) = x^2 + y^3$ is the potential function. Since \mathbf{F} is conservative, the actual path of C is not relevant. We just need its two endpoints. When $t = -1$, we have $\mathbf{r}(-1) = \langle (-1)^2, 5(-1) \rangle = \langle 1, -5 \rangle$, and when $t = 3$, we have $\mathbf{r}(3) = \langle (3)^2, 5(3) \rangle = \langle 9, 15 \rangle$. Strictly speaking, $\langle 1, -5 \rangle$ and $\langle 9, 15 \rangle$ are vectors, but if their feet are placed at the origin, then their heads point to the ordered pairs $(1, -5)$ and $(9, 15)$. In this way, a point as an ordered pair can be inferred from a vector.

Therefore, we have

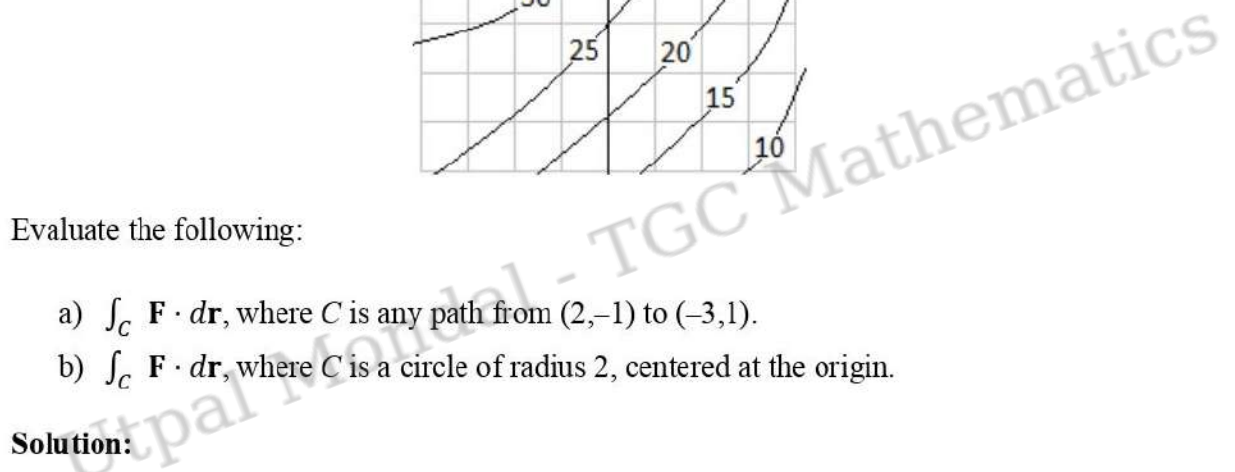
$$\int_C \mathbf{F} \cdot d\mathbf{r} = [x^2 + y^3]_{(1,-5)}^{(9,15)} = ((9)^2 + (15)^3) - ((1)^2 + (-5)^3) = 3580.$$



Evaluate the following:

- $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is any path from $(2, -1)$ to $(-3, 1)$.
- $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is a circle of radius 2, centered at the origin.

Solution:



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Solution:

16.3 The Fundamental Theorem of Line Integrals

Recall the Fundamental Theorem of Calculus for a single-variable function f :

$$\int_a^b f'(x) \, dx = f(b) - f(a)$$

It says that we may evaluate the integral of a derivative simply by knowing the values of the function at the endpoints of the interval of integration $[a, b]$.

The Fundamental Theorem of Line Integrals is a precise analogue of this for multi-variable functions. The primary change is that *gradient* ∇f takes the place of the derivative f' in the original theorem.

Theorem (Fundamental Theorem of Line Integrals). *Suppose that C is a smooth curve from points A to B parameterized by $\mathbf{r}(t)$ for $a \leq t \leq b$. Let f be a differentiable function whose domain includes C and whose gradient vector ∇f is continuous on C . Then*

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) = f(B) - f(A)$$

The long caveats about differentiability and continuity are merely so that the original Fundamental Theorem of Calculus can be invoked in the proof.

Proof. ($n = 2$ or 3 for the purposes of this course)

$$\begin{aligned} \int_C \nabla f \cdot d\mathbf{r} &= \int_C \begin{pmatrix} f_{x_1} \\ \vdots \\ f_{x_n} \end{pmatrix} \cdot \begin{pmatrix} dx_1 \\ \vdots \\ dx_n \end{pmatrix} = \int_C \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n \\ &= \int_a^b \left(\frac{\partial f}{\partial x_1} \frac{dx_1}{dt} + \cdots + \frac{\partial f}{\partial x_n} \frac{dx_n}{dt} \right) dt \\ &= \int_a^b \frac{d}{dt} f(x_1(t), \dots, x_n(t)) dt = \int_a^b \frac{d}{dt} f(\mathbf{r}(t)) dt \quad (\text{chain rule}) \\ &= f(\mathbf{r}(b)) - f(\mathbf{r}(a)) \end{aligned}$$

where we applied FTC in the final step. ■

The Theorem can be alternatively stated: if \mathbf{F} is a conservative vector field with potential function f then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(\text{end of } C) - f(\text{start of } C)$$

We say that a line integral in a conservative vector field is *independent of path*.

Examples

1. Let C be the curve parameterized by $\mathbf{r}(t) = \begin{pmatrix} 1+\sin^2 t \\ t+\sin t \end{pmatrix}$ for $0 \leq t \leq 2\pi$. Then

$$\int_C \nabla(x^2 y^3) \cdot d\mathbf{r} = x^2 y^3 \Big|_{(1,0)}^{(1,2\pi)} = 8\pi^3 - 0 = 8\pi^3$$

2. Let C be the curve parameterized by $\mathbf{r}(t) = \begin{pmatrix} t^3-1 \\ t-t^{-1} \end{pmatrix}$ for $2 \leq t \leq 3$. Then

$$\int_C \sin y \, dx + x \cos y \, dy = \int_C \nabla(x \sin y) \cdot d\mathbf{r} = x \sin y \Big|_{(7, \frac{3}{2})}^{(26, \frac{8}{3})} = 26 \sin \frac{8}{3} - 7 \sin \frac{3}{2}$$

3. Evaluate the line integrals $\int_{C_i} y \, dx + x \, dy$ where C_1 is the straight line from $(0,0)$ and $(1,1)$, and C_2 is the parabola $y = x^2$ between the same points.

For the first curve we have $\mathbf{r}(t) = \begin{pmatrix} t \\ t \end{pmatrix}$, so

$$\int_{C_1} y \, dx + x \, dy = \int_0^1 2t \, dt = 1$$

For the second curve we have $\mathbf{r}(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}$, so

$$\int_{C_2} y \, dx + x \, dy = \int_0^1 t^2 \, dt + 2t^2 \, dt = 1$$

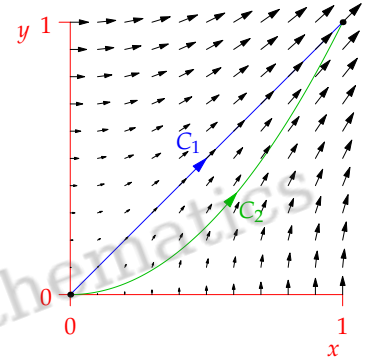
We expected the two solutions to be the same since $\begin{pmatrix} y \\ x \end{pmatrix} = \nabla(xy)$ is conservative. We could simply have applied the Fundamental Theorem:

$$\int_{C_i} \begin{pmatrix} y \\ x \end{pmatrix} \cdot d\mathbf{r} = \int_{C_i} \nabla(xy) \cdot d\mathbf{r} = xy \Big|_{(0,0)}^{(1,1)} = 1 - 0 = 1$$

4. Evaluate $\int_C y^2 z \, dx + 2xyz \, dy + xy^2 \, dz$ along any curve joining $(1,0,0)$ and $(2,1,-1)$.

The integral is $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = \nabla(xy^2z)$, so the path is irrelevant and we obtain

$$\int_C y^2 z \, dx + 2xyz \, dy + xy^2 \, dz = xy^2 z \Big|_{(1,0,0)}^{(2,1,-1)} = -2$$



Conservation of Energy The terminology (conservative, potential, etc.) all comes from Physics.

There are two primary forms of energy: potential (stored) and kinetic (motion).

Suppose that a particle of mass m follows a curve C through a conservative force field $\mathbf{F} = -\nabla f$.

We parameterize the curve so that the particle is at position $\mathbf{r}(t)$ at time t . Its *velocity vector* is then

$$\mathbf{v}(t) = \mathbf{r}'(t)$$

The particle has *kinetic energy* $K = \frac{1}{2}m|\mathbf{v}|^2$ and is said to have *potential energy* f .

Now we evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ in two ways.

1. Newton's second law ($\mathbf{F} = m\mathbf{a} = m\mathbf{v}'$) says that¹

$$\int_C \mathbf{F} \cdot d\mathbf{r} = m \int_{t_0}^{t_1} \mathbf{v}'(t) \cdot \mathbf{v}(t) \, dt = m \int_{t_0}^{t_1} \frac{d}{dt} \frac{1}{2} |\mathbf{v}|^2 \, dt = \frac{1}{2} m |\mathbf{v}|^2 \Big|_{t_0}^{t_1} = \Delta K$$

is the change in kinetic energy over the path.

¹By the product rule, $\frac{d}{dt} \mathbf{v} \cdot \mathbf{v} = \mathbf{v}' \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v}' = 2\mathbf{v}' \cdot \mathbf{v} = 2|\mathbf{v}|^2$.

2. Alternatively we may use the Fundamental Theorem:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = - \int_C \nabla f \cdot d\mathbf{r} = -f(\mathbf{r}(t)) \Big|_{t_0}^{t_1} = -\Delta f$$

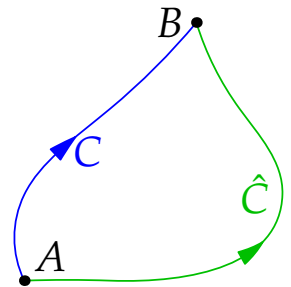
is negative the change in potential energy of the particle over the path.

Therefore $\Delta f + \Delta K = 0$, and so total energy is conserved. Since Physicists always want energy to be conserved, they typically choose potential functions to have a *negative sign*: $\mathbf{F} = -\nabla f$. In mathematics, we omit the negative.

Path Independence

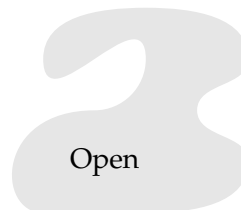
The Fundamental Theorem has the amazing interpretation that line integrals in conservative vector fields depend only on a curve's endpoints. We want to turn this idea on its head. Is it the case that a line integral (or integrals?) being independent of path forces a vector field to be conservative?

Definition. A line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path if $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{\hat{C}} \mathbf{F} \cdot d\mathbf{r}$ for any curve \hat{C} with the same endpoints as C

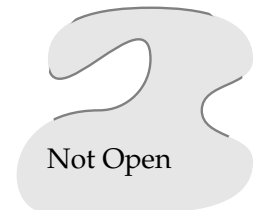


Before we can state the relevant theorems, we need to understand the meaning of several terms.

Definition. A region D is open if it contains no boundary points.



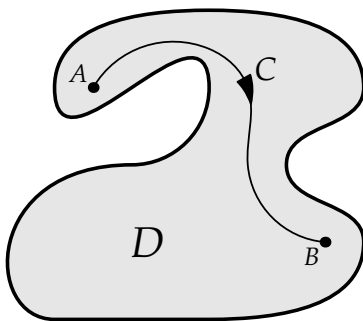
Open



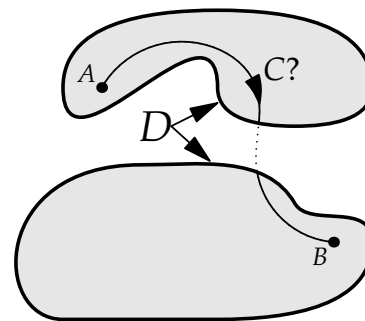
Not Open

For example, the inside of the unit disk $D = \{(x, y) : x^2 + y^2 < 1\}$ is open.

Definition. A region D is (path-)connected if every pair of points A, B in D can be joined by a curve lying entirely in D .



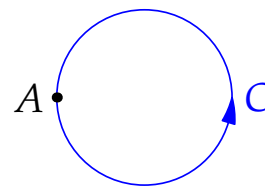
A connected region with curve C joining A, B



A disconnected region: cannot join A, B with a curve lying in D .

The adjectives *open* and *connected* apply only to *domains/regions* in this course. The final adjective applies only to *curves*.

Definition. A curve is closed if it starts and finishes at the same point.



Independence of Path and Closed Curves The following important Theorem relates being independent of path to line integrals round closed curves.

Theorem. Let C be a curve in a connected region D . Then $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path if and only if $\int_S \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path S in D .

Read the Theorem carefully: if even one curve C has $\int_C \mathbf{F} \cdot d\mathbf{r}$ independent of path, then the line integrals over *all* curves must be independent of path. This says that independence of path is really a property of the *vector field* \mathbf{F} rather than a specific curve.

Proof. Suppose first that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path and let S be a closed curve in D . Since D is connected, we may join the starting point A of C to some fixed point P on S by a curve J lying in D . Writing $-J$ for the same curve travelled in reverse, we see that the composite curve

$$C_1 = J \cup S \cup (-J) \cup C$$

has the same endpoints as C . By independence of path, it follows that

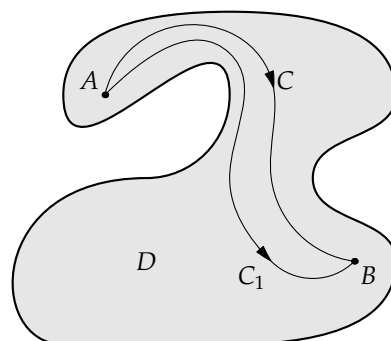
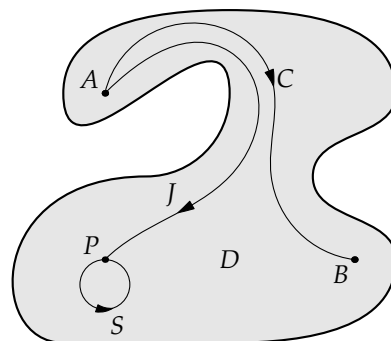
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \left[\int_J + \int_S + \int_{-J} + \int_C \right] \mathbf{F} \cdot d\mathbf{r}$$

However $\int_{-J} \mathbf{F} \cdot d\mathbf{r} = -\int_J \mathbf{F} \cdot d\mathbf{r}$, since travelling in reverse changes the sign of a work integral. Cancelling these and the $\int_C \mathbf{F} \cdot d\mathbf{r}$ terms from both sides forces us to conclude that $\int_S \mathbf{F} \cdot d\mathbf{r} = 0$.

Conversely, suppose that the line integral round any closed curve S evaluates to zero. Let C and C_1 be two curves with the same endpoints. Then $C \cup (-C_1)$ is a closed curve S , whence

$$\int_C \mathbf{F} \cdot d\mathbf{r} - \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = 0.$$

Thus $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path, regardless of C . ■



Finally, we see that independence of path is equivalent to the vector field being conservative.

Theorem. Let \mathbf{F} be a continuous vector field on, and C a curve in, an open, connected region. Then $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path if and only if \mathbf{F} is conservative.

Proof of Theorem (for 2 dimensions). If \mathbf{F} is conservative with potential function f then the Fundamental Theorem tells us that $\int_C \mathbf{F} \cdot d\mathbf{r}$ depends only on the endpoints of C .

Conversely, suppose that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path. By the previous Theorem, $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path for *any* curve C in D . Choose a point A and define $f(x, y) = \int_C \mathbf{F} \cdot d\mathbf{r}$ where C is any curve joining A to (x, y) . The function f is well-defined because $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path. We claim that f is a potential function for \mathbf{F} .

Choose $(x, y) \in D$. Since D is open, there exists a point $(x_1, y) \in D$ such that $x_1 < x$. Let C_1 be a path from A to (x_1, y) and C_2 the line segment thence to (x, y) . Then

$$f(x, y) = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

Since x_1 is constant, the first integral is independent of x and so

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

Now let $\mathbf{F} = \begin{pmatrix} P \\ Q \end{pmatrix}$. Along the curve C_2 we have y constant, hence $dy = 0$. Therefore

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \int_{(x_1, y)}^{(x, y)} P dx + Q dy = \frac{\partial}{\partial x} \int_{(x_1, y)}^{(x, y)} P dx = \frac{d}{dx} \int_{x_1}^x P(t, y) dt = P(x, y)$$

by the Fundamental Theorem of Calculus.

A similar argument (choose $(x, y_1) \in D$ with $y_1 < y$) shows that $Q(x, y) = \frac{\partial f}{\partial y}$. Putting this together we see that $\nabla f = \mathbf{F}$ and so \mathbf{F} is conservative. ■

The proof in three dimensions requires a similar third argument for $\frac{\partial f}{\partial z}$.

Example Evaluate the line integrals $\int_{C_i} \left(\frac{2y}{x} \right) \cdot d\mathbf{r}$ over the same line and parabola as before.

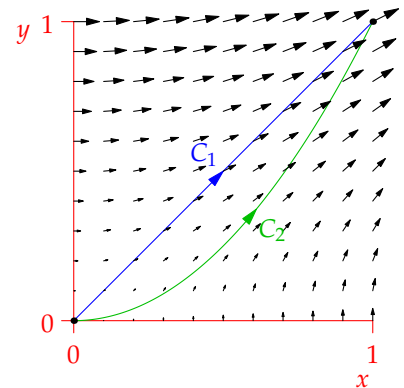
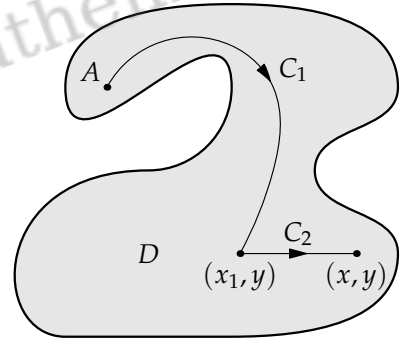
For the first curve we have $\mathbf{r}(t) = \begin{pmatrix} t \\ t \end{pmatrix}$, whence

$$\int_{C_1} 2y dx + x dy = \int_0^1 3t dt = \frac{3}{2}$$

For the second curve we have $\mathbf{r}(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}$, and so

$$\int_{C_2} 2y dx + x dy = \int_0^1 2t^2 dt + \int_0^1 2t^2 dt = \frac{4}{3} \neq \frac{3}{2}$$

Considering the strength of the arrows in the picture it should be clear why $\int_{C_2} < \int_{C_1}$. The fact that these integrals give different values tells us that $\mathbf{F} = \left(\frac{2y}{x} \right)$ is *not* conservative.



When is \mathbf{F} conservative? Searching for a potential function can involve a lot of work. It is useful to know if a vector field is conservative *before* you start looking.

Theorem. Suppose that P, Q have continuous first derivatives on a region D , then

$$\mathbf{F} = \begin{pmatrix} P \\ Q \end{pmatrix} \text{ conservative} \implies \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \text{ throughout } D.$$

$$\text{Equivalently: } \frac{\partial Q}{\partial x} \neq \frac{\partial P}{\partial y} \implies \mathbf{F} = \begin{pmatrix} P \\ Q \end{pmatrix} \text{ not conservative.}$$

Proof. Suppose $\mathbf{F} = \nabla f = \begin{pmatrix} f_x \\ f_y \end{pmatrix}$. Then $f_{xy} = f_{yx}$, hence result. ■

While the Theorem can easily tell us that a vector field is *not* conservative, it doesn't quite work in reverse. We first need some more topology.

Simply-connected regions

Definition. A connected region D is simply-connected if and only if every closed curve in D may be shrunk to a point without leaving D .

D simply-connected

D connected but not simply-connected: cannot shrink C to a point due to the 'hole' in D

Theorem (Proof requires Green's Theorem). Suppose $\mathbf{F} = \begin{pmatrix} P \\ Q \end{pmatrix}$ has continuous partial derivatives on a simply-connected region D and suppose that P, Q have continuous partial derivatives. Then \mathbf{F} is conservative if and only if $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ throughout D .

Examples

1. Find a potential function, if there is one, for the vector field on $\mathbf{F} = \begin{pmatrix} 2xy \\ x^2 \end{pmatrix}$ on $D = \mathbb{R}^2$.

You can dive straight in to computing, but it is helpful to use the Theorem first so that you don't waste time.

$$\frac{\partial Q}{\partial x} = 2x = \frac{\partial P}{\partial y} \implies \mathbf{F} \text{ is conservative.}$$

Now that we know a potential function f exists, we can solve for it:

$$f_x = P = 2xy \implies f(x, y) = x^2y + g(y)$$

$$f_y = Q = x^2 \implies f(x, y) = x^2y + h(x)$$

for unknown functions g, h . Choosing $g(y) = h(x) = c$ (constant) yields all possible potential functions $f(x, y) = x^2y + c$.

2. $\mathbf{F}(x, y) = \begin{pmatrix} y \sin x \\ x \sin x \end{pmatrix}$ has domain $D = \mathbb{R}^2$. We quickly see that

$$\frac{\partial Q}{\partial x} = \sin x + x \cos x \quad \text{and} \quad \frac{\partial P}{\partial y} = \sin x$$

$$\frac{\partial Q}{\partial x} \neq \frac{\partial P}{\partial y} \implies \mathbf{F} \text{ is non-conservative.}$$

3. Let $\mathbf{F}(x, y) = \begin{pmatrix} (2x+x^2y)e^{xy} \\ x^3e^{xy} \end{pmatrix}$. Prove that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ where C is any closed curve in the plane

We calculate the mixed partial derivatives:

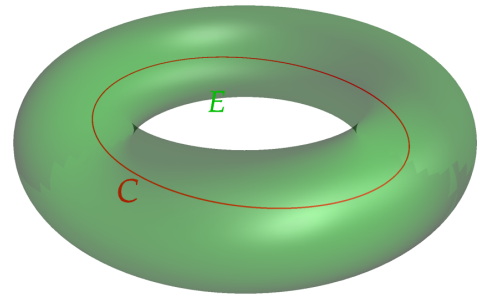
$$\frac{\partial}{\partial x} x^3 e^{xy} = 3x^2 e^{xy} + x^3 y e^{xy} = x^2 e^{xy} (3 + xy)$$

$$\frac{\partial}{\partial y} (2x + x^2 y) e^{xy} = x^2 e^{xy} + (2x + x^2 y) x e^{xy} = x^2 e^{xy} (3 + xy)$$

These are equal, hence \mathbf{F} is conservative and consequently all line integrals over closed paths C evaluate to zero.

What changes in three dimensions? Essentially nothing! A connected volume E is still simply-connected if every curve can be shrunk to a point without leaving E .

The interior of a solid torus E is non-simply-connected since a curve can be drawn inside it which cannot be shrunk to a point



The primary difference is that the corresponding Theorem is not quite as easy to use as the 2D version...

Theorem. Suppose $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on a simply-connected region $E \subseteq \mathbb{R}^3$ and suppose that P, Q, R have continuous partial derivatives. Then \mathbf{F} is conservative iff

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}, \quad \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}, \quad \text{and} \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \quad \text{throughout } E.$$

The proof requires Stokes' Theorem and will be much easier to remember once we've introduced Curl. At the present it is not worth remembering.

Example Let $\mathbf{F}(x, y) = \begin{pmatrix} 2x+2y+6xyz+z \\ 2x+3x^2z \\ 3x^2y+x \end{pmatrix}$. Writing $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ we can laboriously check that $P_y = Q_x$, etc., to see that \mathbf{F} is indeed conservative.

Writing $\mathbf{F} = \nabla f$ we can partially integrate:

$$\begin{aligned} f_x &= P = 2x + 2y + 6xyz + z \\ \implies f(x, y, z) &= x^2 + 2xy + 3x^2yz + xz + g(y, z) \\ f_y &= Q = 2x + 3x^2z \\ \implies f(x, y, z) &= 2xy + 3x^2yz + h(x, z) \\ f_z &= R = 3x^2y + x \\ \implies f(x, y, z) &= 3x^2yz + xz + j(x, y) \end{aligned}$$

for unknown functions g, h, j . Suitable choices yield

$$f(x, y, z) = x^2 + 2xy + 3x^2yz + xz$$

Does Simple-Connectedness Really Matter? Let us analyze the vector field $\mathbf{F} = \frac{1}{x^2 + y^2} \begin{pmatrix} -y \\ x \end{pmatrix}$ and try to decide if it is conservative.

1. First compute the partial derivatives:

$$\begin{aligned} \frac{\partial Q}{\partial x} &= \frac{\partial}{\partial x} \frac{x}{x^2 + y^2} = \frac{1}{x^2 + y^2} - \frac{2x^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2} \\ \frac{\partial P}{\partial y} &= \frac{\partial}{\partial y} \frac{-y}{x^2 + y^2} = \frac{-1}{x^2 + y^2} + \frac{2y^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2} \end{aligned}$$

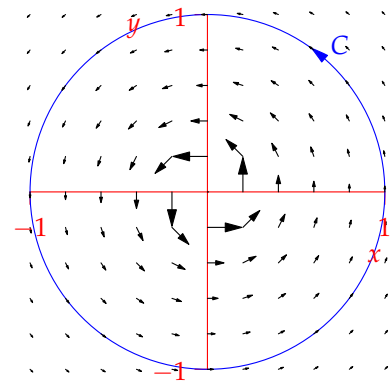
Since $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ we want to say that \mathbf{F} is conservative.

2. Now calculate the line integral of \mathbf{F} around the unit circle:

$$\mathbf{r}(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}, \quad 0 \leq t \leq 2\pi$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} dt = 2\pi$$

Since $\int_C \mathbf{F} \cdot d\mathbf{r} \neq 0$ we conclude that \mathbf{F} is non-conservative.



At least one of these arguments has to be false, but which? The answer, strangely, depends on your choice of *domain*.

- With no additional information, you should assume that the domain of the vector field is the largest set possible, in this case the punctured plane $D = \mathbb{R}^2 \setminus \{(0,0)\}$, the plane with the

origin removed. This is non-simply-connected; indeed the unit circle cannot be shrunk to a point without some part of it passing through the origin! The easy $Q_x = P_y$ theorem does not apply and argument 1. is false. The vector field is conservative.

- Suppose that the domain was restricted so that it was simply connected. For instance, we could exclude the positive x -axis and choose the domain

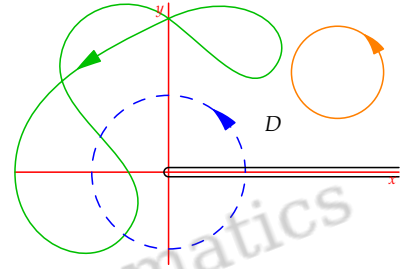
$$D = \mathbb{R}^2 \setminus \{(x, 0) : x \geq 0\}$$

Now D is simply-connected and \mathbf{F} is conservative. Moreover $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed curve in D . Note that the original unit circle is no longer a curve in D , whence argument 2. is now false.

In the picture, the line integrals round the solid curves are zero, while around the blue curve (which is not in D) the line integral evaluates to 2π . You can easily check that a suitable potential function for \mathbf{F} on D is $f = \theta$ in polar-coordinates; otherwise said,

$$f(x, y) = \begin{cases} \tan^{-1} \frac{y}{x} & y > 0 \\ \pi & y = 0 \\ \pi + \tan^{-1} \frac{y}{x} & y < 0 \end{cases}$$

The problem in extending this potential function to the entire punctured plane is that θ is not continuous everywhere; typically it is discontinuous on the positive x -axis.



Summary If $\mathbf{F}(x, y) = P\mathbf{i} + Q\mathbf{j}$ is defined on a connected region $D \subseteq \mathbb{R}^2$, where P, Q have continuous first derivatives then the following are equivalent:

1. \mathbf{F} is conservative
2. \mathbf{F} is a gradient field ($= \nabla f$ for some potential function f)
3. $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path for any curve C in D
4. $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every piecewise smooth closed curve C in D

In addition, if D is simply-connected we also have

$$5. \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ on a volume $E \subseteq \mathbb{R}^3$ then the fifth condition becomes

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}, \quad \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}$$

Green's Theorem, Surface Integrals, Stoke's Theorem, Divergence Theorem

1 Green's Theorem

A curve C in \mathbb{R}^2 or \mathbb{R}^3 is *simple* if it does not intersect itself. It is closed if it loops back around on itself. It is piecewise smooth if a parametrization $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ of C has a continuous nonzero derivative $\mathbf{r}'(t)$ at all but finitely many points.

Theorem 1.1 (*Green's Theorem, Line Integral Form*)

Let C be a closed piecewise smooth curve in \mathbb{R}^2 with counterclockwise orientation and let D be the region bounded by C . Let $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$ be a vector field in \mathbb{R}^2 so that $P(x, y)$ and $Q(x, y)$ have continuous partials in an open region containing D . Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Technical Comment: The components P and Q of $\mathbf{F}(x, y)$ above *must* have continuous partials on an open set containing the region D . Otherwise Green's Theorem may not hold!

If $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$ is a vector field in \mathbb{R}^2 and C is a curve parametrized by $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, then a normal vector to the curve is given by $\mathbf{n}(t) = \langle y(t), -x(t) \rangle$. If you are walking along C in the direction given by $\mathbf{r}(t)$, then $\mathbf{n}(t)$ will point to your right. If C is a simple closed curve bounding a region D and $\mathbf{r}(t)$ traverses the curve in the counterclockwise direction then $\mathbf{n}(t)$ points away from D . $\mathbf{u}(t) = \frac{\mathbf{n}(t)}{|\mathbf{n}(t)|}$ gives a unit normal to C .

Definition 1.2 (*Flux*)

If $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$ is a vector field in \mathbb{R}^2 and C is a curve (not necessarily closed), then the flux of \mathbf{F} through C in the direction of \mathbf{u} is

$$\int_C \mathbf{F} \cdot \mathbf{u} ds$$

The following version of Green's Theorem will also be referred to as the Divergence Theorem. If you are asked to use the divergence theorem on a planar region, this is the version you need to use.

Theorem 1.3 (*Green's Theorem, Divergence form*)

Same hypotheses as in the Line Integral form of Green's theorem. Then

$$\int_C \mathbf{F} \cdot \mathbf{u} ds = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA = \iint_D \operatorname{div} \mathbf{F} dA$$

2 Surface Integrals

Let $S \subset \mathbb{R}^3$ be a surface parametrized by $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$, where u and v come from the domain D in the u - v plane. $\mathbf{n}(u, v) = \mathbf{r}_u \times \mathbf{r}_v$ is a normal vector to the surface. If $f(x, y, z)$ is a function of three variables, then the surface integral $\int_S f dS$ is evaluated by

$$\begin{aligned} \iint_S f dS &= \iint_D f(x(u, v), y(u, v), z(u, v)) |\mathbf{r}_u \times \mathbf{r}_v| dA \\ &= \iint_D f(x(u, v), y(u, v), z(u, v)) |\mathbf{n}(u, v)| dA \end{aligned}$$

$\mathbf{u} = \frac{\mathbf{n}}{|\mathbf{n}|}$ is a unit normal to the surface. If $\mathbf{F}(x, y, z) = \langle F_1(x, y, z), F_2(x, y, z), F_3(x, y, z) \rangle$ is a vector field in \mathbb{R}^3 then the flux of \mathbf{F} through S in the direction of \mathbf{u} is the surface integral

$$\begin{aligned} \int \int_S \mathbf{F} \cdot \mathbf{u} \, dS &= \int \int_D \mathbf{F}(x(u, v), y(u, v), z(u, v)) \cdot \frac{\mathbf{n}}{|\mathbf{n}|} |\mathbf{n}| \, dA \\ &= \int \int_D \mathbf{F}(x(u, v), y(u, v), z(u, v)) \cdot \mathbf{n}(u, v) \, dA \end{aligned}$$

3 Stoke's Theorem

An *orientation* of a surface S in \mathbb{R}^3 is a choice of unit normal \mathbf{u} to the surface (can only point two directions). An orientation of a curve in \mathbb{R}^3 is a choice of direction to travel on C (in \mathbb{R}^2 this is either clockwise or counterclockwise). If S is a surface in \mathbb{R}^3 oriented by a unit normal \mathbf{u} and $C = \partial S$ is the boundary of S then \mathbf{u} induces an orientation on C by the following rule: Stand on the boundary of S with the surface on your left hand side and your head pointing in the direction of \mathbf{u} . The direction in which you are facing is the direction of the induced orientation on C . So when you are walking along C in the direction of the induced orientation with your head pointing in the direction of the normal \mathbf{u} , the surface is always on your left.

Theorem 3.1 (Stoke's Theorem)

Let $S \subset \mathbb{R}^3$ be a piecewise smooth surface parametrized by $\mathbf{r}(u, v)$ with a simple, closed, piecewise smooth boundary $\partial S = C$ and let $\mathbf{F}(x, y, z) = \langle F_1(x, y, z), F_2(x, y, z), F_3(x, y, z) \rangle$ be a vector field in \mathbb{R}^3 so that F_1, F_2 , and F_3 have continuous partials in an open region containing S . If S is oriented by the unit normal \mathbf{u} and C has the induced orientation then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int \int_S (\text{curl } \mathbf{F}) \cdot \mathbf{u} \, dS$$

4 Divergence Theorem (3 dimensional form)

If S is the boundary of a solid region E (for instance if S is the surface of a cube or a ball or pyramid or any other similar solid geometric object E) then the *positive* orientation on S is the one given by a unit normal \mathbf{u} which points outward (away from E).

Theorem 4.1 (Divergence Theorem - 3 dimensional form)

Let E be a simple solid region in \mathbb{R}^3 and $S = \partial E$ the boundary of E with positive orientation. Let $\mathbf{F} = \langle F_1(x, y, z), F_2(x, y, z), F_3(x, y, z) \rangle$ be a vector field in \mathbb{R}^3 with F_1, F_2 , and F_3 having continuous partials on an open region containing E . Then

$$\int \int_S \mathbf{F} \cdot \mathbf{u} \, dS = \int \int \int_E \text{div } \mathbf{F} \, dV$$

Note: Remember the Divergence Theorem may also refer to the divergence form of Green's theorem in section 1. Use the divergence theorem of section 1 to evaluate flux of a two dimensional vector field through a simple closed curve in \mathbb{R}^2 . Use the Divergence Theorem above to evaluate flux of a three dimensional vector field through the boundary of a simple solid region in \mathbb{R}^3 .

Courtesy : ---

1. Green's Theorem, Surface Integrals, Stoke's Theorem, Divergence Theorem :
<https://faculty.math.illinois.edu/~dipasqu1/VectorReview.pdf>

2. The Fundamental Theorem of Line Integrals :
<https://www.math.uci.edu/~ndonalds/math2e/16-3fundthm.pdf>

3. Conservative Vector Fields :
https://math.la.asu.edu/~surgent/mat267/examples/Cons_Vec_Fields.pdf

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