Laplace's Equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

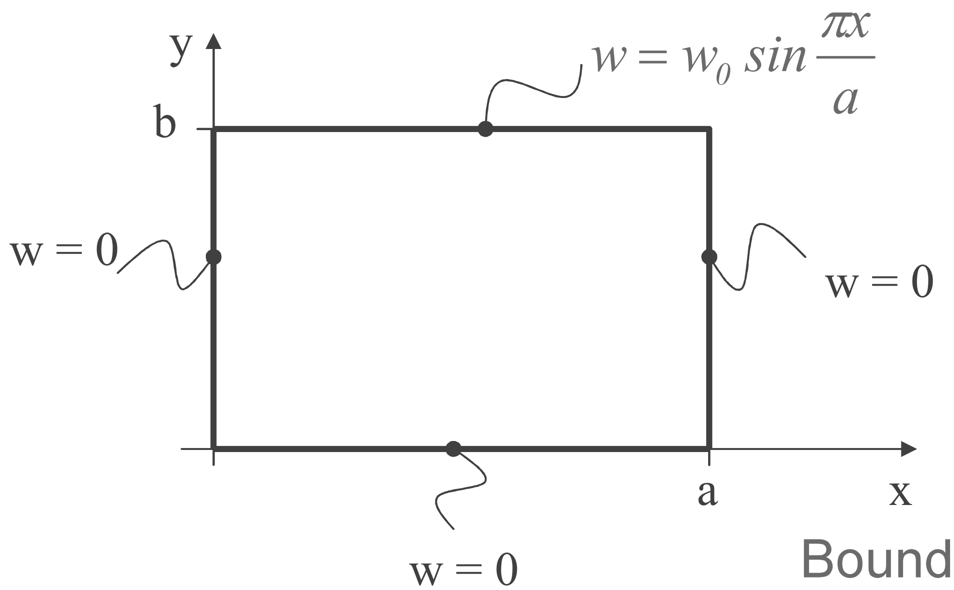
In the vector calculus course, this appears as $\nabla^2 \phi = 0$ where $\nabla = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix}$

Note that the equation has **no** dependence on time, just on the spatial variables x,y. This means that Laplace's Equation describes **steady state** situations such as:

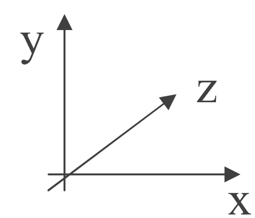
- steady state temperature distributions
- steady state stress distributions
- steady state potential distributions (it is also called the *potential equation*
- steady state flows, for example in a cylinder, around a corner, …

Stress analysis example: Dirichlet conditions

Steady state stress analysis problem, which satisfies Laplace's equation; that is, a stretched elastic membrane on a rectangular former that has prescribed out-of-plane displacements along the boundaries



w(x,y) is the displacement in z-direction



To solve:

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0$$

Boundary conditions

$w(0, y) = 0,$ for $0 \le y \le b$

$w(x,0) = 0,$ for $0 \le x \le a$

$w(a, y) = 0,$ for $0 \le y \le b$

$w(x,b) = w_0 \sin \frac{\pi}{a} x,$ for $0 \le x \le a$

Solution by separation of variables

$$w(x, y) = X(x)Y(y)$$

from which $$X''Y + XY'' = 0$$

and so $$\frac{X''}{X} + \frac{Y''}{Y} = 0$$

as usual … $$\frac{X''}{X} = -\frac{Y''}{Y} = k$$

where *k* is a constant that is either equal to, >, or < 0.

Case *k=0*

$$X(x) = (Ax + B), Y(y) = (Cy + D)$$

$w(0, y) = 0 \Rightarrow B = 0$ or $C = D = 0$

if $C = D = 0$, then $Y(y) \equiv 0$, so $w(x, y) \equiv 0$

Continue with $B = 0$:$w(x, y) = Ax(Cy + D)$

$w(x,0) = 0 \Rightarrow ADx = 0$

Either $A = 0$ (so $w \equiv 0$) or $D = 0$

Continue with $w(x, y) = ACxy$

$w(a, y) = 0 \Rightarrow ACay = 0 \Rightarrow A = 0$ or $C = 0 \Rightarrow w(x, y) \equiv 0$

That is, the case *k=0* is not possible

Case *k>0*

Suppose that $k = \alpha^2$, so that

$$w(x, y) = (A\cosh\alpha x + B\sinh\alpha x)(C\cos\alpha y + D\sin\alpha y)$$

Recall that $\cosh 0 = 1, \sinh 0 = 0$

$w(0, y) = 0 \Rightarrow A(C\cos\alpha y + D\sin\alpha y) = 0$

$C = D = 0 \Rightarrow w(x, y) \equiv 0$

Continue with $A = 0 \Rightarrow w(x, y) = B\sinh\alpha x(C\cos\alpha y + D\sin\alpha y)$

$w(x,0) = 0 \Rightarrow BC\sinh\alpha x = 0$

$B = 0 \Rightarrow w(x, y) \equiv 0$

Continue with $C = 0 \Rightarrow w(x, y) = BD\sinh\alpha x\sin\alpha y$

$w(a, y) = 0 \Rightarrow BD\sinh\alpha a\sin\alpha y = 0$

so either $B = 0$ or $D = 0 \Rightarrow w(x, y) \equiv 0$

Again, we find that the case *k>0* is not possible

Final case *k<0*

Suppose that $k = -\alpha^2$

$$w(x, y) = (A\cos\alpha x + B\sin\alpha x)(C\cosh\alpha y + D\sinh\alpha y)$$

$w(0, y) = 0 \Rightarrow A(C\cosh\alpha y + D\sinh\alpha y) = 0$

as usual, $C = D = 0 \Rightarrow w \equiv 0$

continue with $A = 0 \Rightarrow w(x, y) = B\sin\alpha x(C\cosh\alpha y + D\sinh\alpha y)$

$w(x,0) = 0 \Rightarrow BC\sin\alpha x = 0$

$B = 0 \Rightarrow w \equiv 0$

continue with $C = 0 \Rightarrow w(x, y) = BD\sin\alpha x\sinh\alpha y$

$w(a, y) = 0 \Rightarrow BD\sin\alpha a\sinh\alpha y = 0$

$B = 0$ or $D = 0 \Rightarrow w \equiv 0$

$$\sin\alpha a = 0 \Rightarrow \alpha = n\frac{\pi}{a} \Rightarrow w_n(x, y) = BD\sin n\frac{\pi}{a}x\sinh n\frac{\pi}{a}y$$

Solution

Applying the first three boundary conditions, we have

$$w(x,y) = \sum_{n=1}^{\infty} K_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi y}{a}$$

The final boundary condition is: $w(x,b) = w_0 \sin\frac{\pi x}{a}$

which gives: $w_0 \sin\frac{\pi x}{a} = \sum_{n=1}^{\infty} K_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi b}{a}$

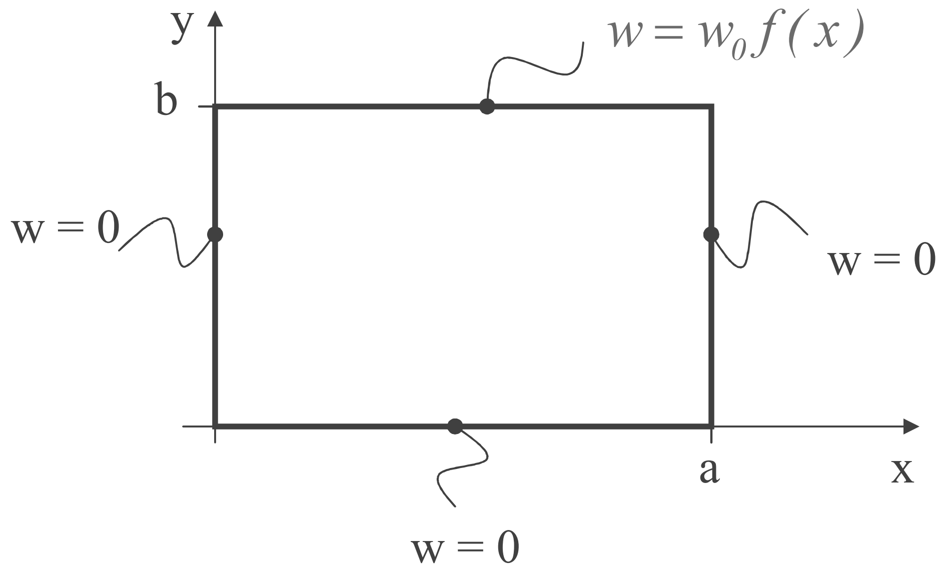
We can see from this that *n* must take only one value, namely 1, so that $K_1 = \frac{w_0}{\sinh\frac{\pi b}{a}}$

and the final solution to the stress distribution is

$$w(x,y) = \frac{w_0}{\sinh\frac{\pi b}{a}} \sin\frac{\pi x}{a} \sinh\frac{\pi y}{a}$$

Check out: http://mathworld.wolfram.com/HyperbolicSine.html

More general boundary condition



Then

$$w_0 f(x) = \sum_{n=1}^{\infty} K_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi b}{a}$$

and as usual we use orthogonality formulae/HLT to find the K_n

Types of boundary condition

1. The *value* $\phi(x, y)$ is specified at each point on the boundary: "Dirichlet conditions"
2. The *derivative normal to the boundary* $\frac{\partial \phi}{\partial \mathbf{n}}(x, y)$ is specified at each point of the boundary: "Neumann conditions"
3. A mixture of type 1 and 2 conditions is specified

Johann Dirichlet (1805-1859)

http://www-gap.dcs.st-and.ac.uk/~history/Mathematicians/Dirichlet.html

Carl Gottfried Neumann (1832 -1925)

http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Neumann_Carl.html